AMT

INTERMEDIATE

SI METRIC EDITION

THE RYERSON MATHEMATICS PROGRAM

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THE RYERSON MATHEMATICS PROGRAM

CORE TEXTS

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Applied Mathematics for Today: Intermediate

This is not a revision, but a new edition of Applied Mathematics for Today Book I by Kierstead et al (1968) in the Ryerson Mathematics Program. Pertinent material from the original companion booklets *Trigonometry* and *Perimeter, Area and Volume* has been incorporated into this second edition.

* *The abbreviations "w." and "mo." are used at points in this text for "week(s)" and "month(s)". These are not SI symbols (since no such symbols exist for these designations) but simply abbreviations.*

Contents

REVIEW AND PREVIEW TO CHAPTER 1

EXERCISE 1

1. Perform the following calculations mentally.

(a) $5 + 11 - 4$
(b) $4 \times 3 - 6$
(c) $3 + 6 \times 2$
(d) $5 + 12 \div 4$
(e) $20 \div 5 + 2$
(f) $3 \times (5 - 1)$
(g) $(4 + 1) \times 6$
(h) $24 \div (4 + 2)$
(i) $13 \times 0 + 5$
(j) $(4 \div 2) \times (6 \div 3)$
(k) 26×100
(l) 41×100
(m) 24×0.1
(n) 63×0.01
(o) $600 \div 20$
(p) $900 \div 300$
(q) $16 \div 0.1$
(r) $25 \div 0.01$
(s) $3\frac{1}{4} - 2\frac{1}{4} + 7$
(t) $\frac{2}{3} \times \frac{2}{7}$
(u) $1\frac{1}{2} \times \frac{1}{2}$
(v) $16 + 0 \div 6$
(w) 40×0.25
(x) 16×0.5
(y) $\frac{2}{3}$ of $\frac{3}{5}$
(z) $(12 - 6) \div (4 - 2)$

EXERCISE 2

1. Multiplication

(a) 573×48
(b) 99×123
(c) 273×66
(d) 22×1076
(e) 763×846
(f) 2473×66
(g) 989×102
(h) 601×307
(i) 7×3876
(j) 57×382
(k) 7632×19
(l) 293×101
(m) 4682×319
(n) $27\,063 \times 4762$
(o) $21\,133 \times 6874$
(p) 3333×55

2. Division

(a) $2516 \div 37$
(b) $1035 \div 23$
(c) $5508 \div 12$
(d) $17\,750 \div 25$
(e) $29\,450 \div 475$
(f) $61\,803 \div 763$
(g) $17\,388 \div 23$
(h) $1312 \div 16$
(i) $4267 \div 24$
(j) $35\,721 \div 33$
(k) $10\,025 \div 163$
(l) $45\,869 \div 1456$
(m) $24\,586 \div 94$
(n) $7889 \div 56$
(o) $13\,725 \div 256$
(p) $20\,002 \div 85$

EXERCISE 3

1. Simplify the following

(a) $3x + 4x - 2x$
(b) $3a + 7a - 5a$
(c) $6m + 4a - 3a + 6m$
(d) $12x - 3x + 2y + x$
(e) $6a - 3b + 2a - 4b$
(f) $11x - 4y - 12x + 3y$
(g) $7ab - 3m + 4m + 6ab$
(h) $6 - 3a + 7b - a + 11$
(i) $12t - 3s + 16s - 14t$
(j) $6xy - 4a - 3a - 7xy$
(k) $21ab - 3bc + 16bc - 11ab$
(l) $3d - 5 - 7d + 6d - 8$
(m) $-3e + 4 - 6e + 3ef + 7$
(n) $2x^2 + 3x - 5 - 7x^2 + 2x - 1$

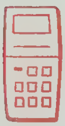
(o) $4a^2 - 3ab + 6ab - 2a^2 + 3ab$
(p) $8x^3 + 3x^2 - 7x - 6x^2 + 3x^3$
(q) $21m^2n - 3mn + 4mn^2 + 6mn - 2m^2n + 3mn^2$
(r) $3abc - 2 - 4ab + 3a + 7abc - 6abc + 7 - 4ab - 7a$
(s) $6ab - 3bc + 6ab - 4ba + 2cb + 3a - 7$

Distributive Property
$a(b + c)$

$= a(b + c)$
$= ab + ac$

2. Expand the following

(a) $2(a + 4)$ (b) $3(a - 2)$ (c) $4(x - 3)$
(d) $2(a - b)$ (e) $7(x + y)$ (f) $8(2x + 1)$
(g) $2(a + 2b + c)$ (h) $6(x - 2y + 7)$
(i) $-2(x - 4)$ (j) $-3(2a - 4b)$
(k) $-5(3a - 2b + c)$ (l) $-(x + 5)$
(m) $10(1 - 3x + 2y)$ (n) $-7(-2 - 3x + 4y)$



Perform the following calculations.

1. $84.76 + 23.78 - 14.73$
2. $8.743 \times 7.681 \times 1.132$
3. $20.42 \times 8.713 \times 0.412$
4. $163.2 \times 22.4 \div 109.4$
5. $843.2 \div 151.7 \times 486.3$
6. $55.46 + 81.73 \times 8.72$
7. $78.46 - 66.51 \times 0.73$
8. $46.32 \times 2.42 + 11.76$
9. $(48.35 + 8.73) \div 5.51$
10. $583.2 \div (16.7 + 857.3)$

Graphing of Data from Investigations

1.1 GRAPHING DATA

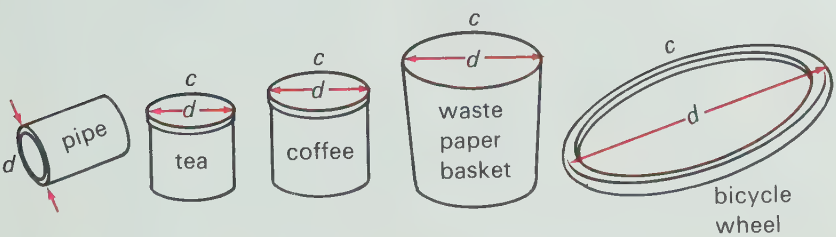
Man has always attempted to interpret the universe in which he exists. Many avenues have been taken to explain his world, and mathematics has played a vital role in this search for knowledge. Experiment, observation, interpretation and logical conclusions are the core of his learning process.

The following section is a series of mathematical investigations which will enable one to collect, to tabulate, and to graph data, and then to draw conclusions. Instructions to be followed in Investigations 1 to 10 have been kept to a minimum so you can develop your own methods.

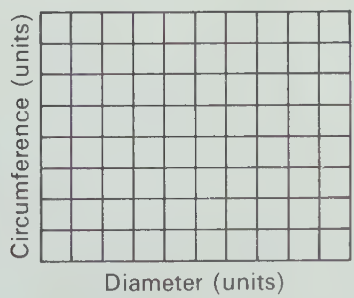
Illustrations, tables, and graphs indicate what must be done.

INVESTIGATION 1.1 *Circles*

Measure the diameter and circumference of circular objects, tabulate, and graph.



Object	Diameter d (units)	Circumference c (units)	$c \div d$
1			
2			
3			
.			
.			
.			
.			
n			

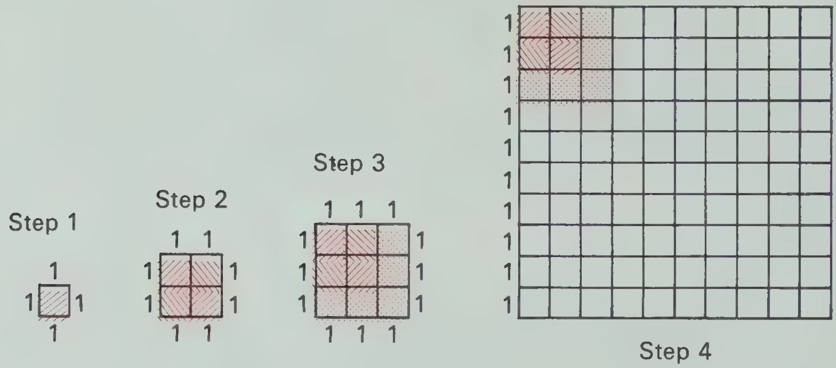


Plot the data from the table on suitable graph paper with scaled axes and join the points. The units should be indicated on the graph.

1. Is the graph a straight line or a curved line?
2. As the diameter increases, does the circumference increase or decrease?
3. If the diameter of a circle is 7 units, what is the circumference of the circle? (Hint: Read your answer from the graph.)
4. Similarly, if the circumference of a circle is 11 units, what is the diameter of the circle?
5. Divide the circumference by the diameter for each case and tabulate.
6. If the diameter is represented by d and circumference by c, write the algebraic equation of their relationship.

INVESTIGATION 1.2 *Squares*

Given a square with sides of unit length, the perimeter is 4 units and the area is 1 square unit.



i.e. Side = 1 unit
Perimeter = 4 units
Area = 1 square unit.

What happens to the volume of a cube when you double the lengths of the edges?

Place three similar squares adjacent to the square in Step 1 to form a larger square as illustrated. The measurements of this larger square are:

Side = 2 units
Perimeter = 8 units
Area = 4 square units.

As in Step 2, place more squares adjacent to the square formed in Step 2. The measurements of this third square are:

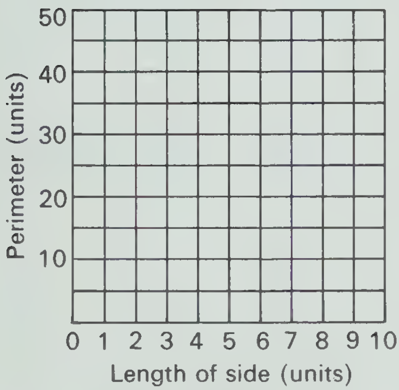
Side = 3 units
Perimeter = 12 units
Area = 9 square units.

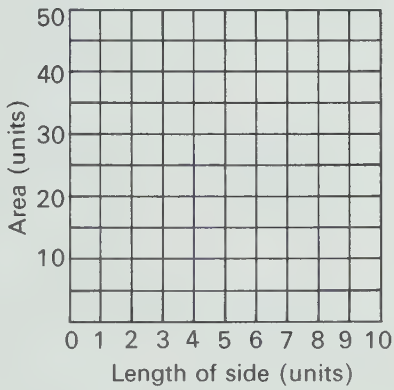
Continue the procedure indicated in the above steps until your square has a side 10 units in length. Measurements:

Side = 10 units
Perimeter = 40 units
Area = 100 square units.

Tabulate and graph this data on two separate graphs: Graph 1—Side and Perimeter; Graph 2—Side and Area.

Length (units)	Perimeter (units)	Area (units)
1	4	1
2	8	4
3	12	9
.	.	.
.	.	.
.	.	.
10	40	100





1. Which of the graphs illustrates:
(a) a linear relation (a straight line)?
(b) a non-linear relation?

2. From your table of values, how is the perimeter related to the length of the sides of the square?

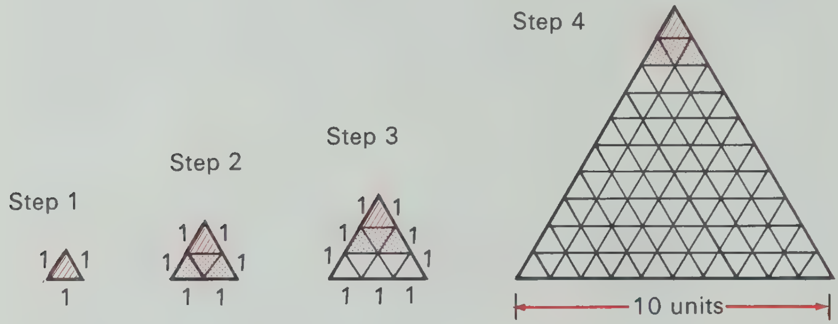
3. If the length is denoted by l and the perimeter by P, write an algebraic equation to represent their relation.

4. Similarly, how is the area in each case related to the length of the sides of each square?

5. If the area is represented by A, write an algebraic equation to represent the relation between area and the length of the side.

INVESTIGATION 1.3 *Equilateral Triangles*

Given an equilateral triangle with sides of unit length, the perimeter is 3 units. We will assume that the area of this triangle is one unit of area.



Side = 1 unit
Perimeter = 3 units
Area = 1 unit of area (assumed).

Place three similar equilateral triangles below the first triangle as illustrated, to form a larger equilateral triangle. The measurements of this larger triangle are:

Side = 2 units
Perimeter = 6 units
Area = 4 units of area

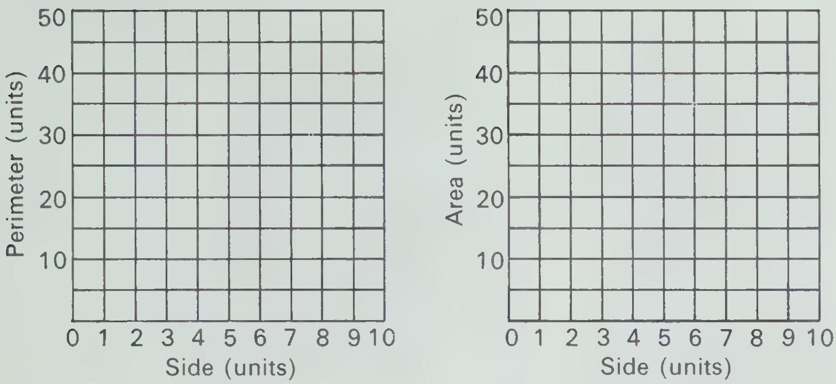
How many three digit numbers can you form using the digits 1, 2, and 3?

As in Step 2, place more equilateral triangles as illustrated, to form a larger triangle. The measurements are:

Side = 3 units
Perimeter = 9 units
Area = 9 units of area

Continue the procedure indicated until your equilateral triangle is made up of at least 10 rows. Tabulate and graph this data.

Side (units)	Perimeter (units)	Area (units)
1	3	1
2	6	4
3	9	9
.	.	.
.	.	.
.	.	.
10	30	100



1. Which of the graphs illustrates:
(a) a linear relation (a straight line)?
(b) a non-linear relation?

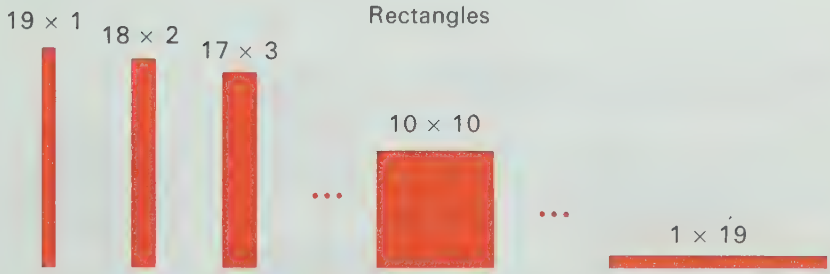
2. From your table of values, how is the perimeter related to the length of the sides of the equilateral triangle?

3. If the length is denoted by l and the perimeter by P, write an algebraic equation to represent their relation.

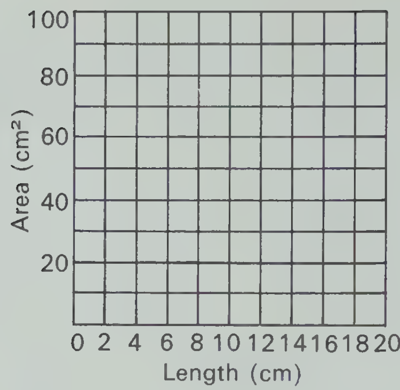
4. If the area is represented by A, write an algebraic equation to represent the relation between area and the length of a side.

INVESTIGATION 1.4 *Rectangles I*

Compare the length of rectangles with the area, if the perimeter is kept constant.



A loop of string 40 cm long is shaped into rectangles as illustrated. The lengths and widths of these rectangles are listed below. Find the area of each rectangle and tabulate your result in the third column. Graph the lengths with the corresponding areas and draw the curve of best fit (a smooth curve which best fits the points).



Length (cm)	Width (cm)	Area (cm^2)
		$A = l \times w$
1	19	
2	18	
3	17	
.	.	
.	.	
.	.	
19	1	

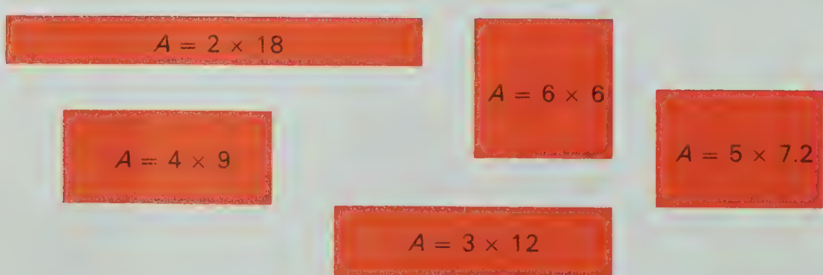
1. For what values of the length is the area:
(a) increasing? (b) decreasing?

2. (a) What length gives the greatest area?
(b) What width gives the greatest area?
(c) Identify the figure.

3. From your graph, give the area for which the length is:
(a) 5.5 cm (b) 14.5 cm (c) 20 cm (d) 0 cm

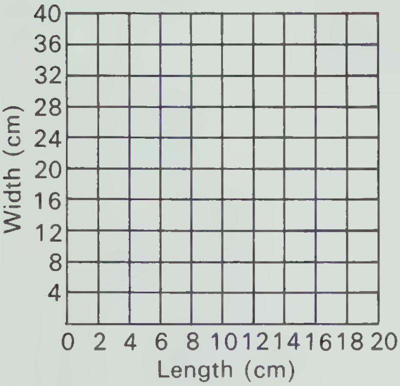
INVESTIGATION 1.5 *Rectangles II*

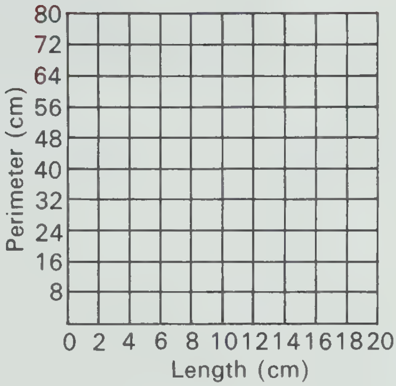
Compare the length of rectangles with the width and perimeter, if the area is kept constant.



For the given lengths, tabulate the widths and perimeters to give an area of 36 cm^2. Graph:
(a) length against width; (b) length against perimeter.

Length (cm)	Width (cm)	Area (cm²)	Perimeter (cm)
1		36	
2		36	
3		36	
.		.	
.		.	
.		.	
10		36	
12		36	
18		36	
1.2		36	
1.8		36	
3.6		36	
7.2		36	





Starting with the word "coin" and changing one letter at a time to form a new word, can you reach "shop" in four changes?

coin
1. —
2. —
3. —
4. shop

1. Do the graphs illustrate linear or non-linear relations?

2. (a) From your graph, find the width when the length is:
(i) 7 cm (ii) 14 cm (iii) 20 cm
(b) From your graph, find the length when the width is:
(i) 11 cm (ii) 16 cm

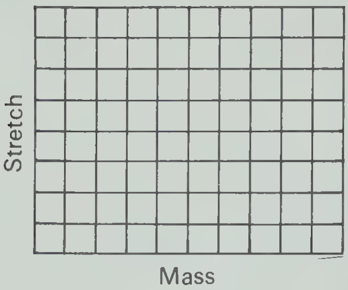
3. As the length increases uniformly, describe the changes in the width.

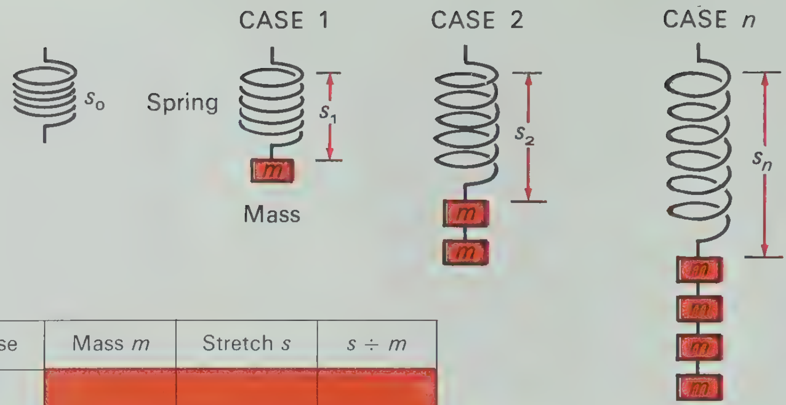
4. From your graph, find the perimeter when the length is:
(a) 15 cm (b) 20 cm (c) 28 cm

5. What is the length when the perimeter is least?

INVESTIGATION 1.6 *Springs*

Measure the stretch of a given spring as the mass on the spring is increased. Tabulate and graph mass against stretch.





Case	Mass m	Stretch s	$s \div m$
1			
2			
3			
.			
.			
.			
n			

1. Is the graph a straight line or a curve?

2. How is the stretch affected when the weight is:
(a) doubled? (b) tripled?

3. From your graph, find the stretch if the masses are:
(a) 1.5 units (b) 2.25 units (c) 3.75 units

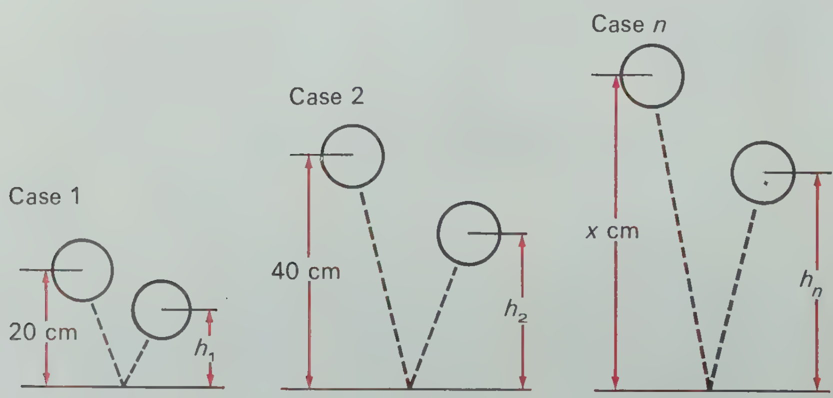
4. Similarly, from your graph you can find the masses of various objects if the stretch is known. Give three examples to illustrate this.

5. From the table of values, divide the stretch by the mass for each case and tabulate. What do you observe?

6. If the stretch is represented by s, and the mass by m, write an algebraic equation of their relationship.

7. Name a practical example of the principle illustrated in this investigation.

INVESTIGATION 1.7 *Bouncing Ball*



Measure the height of the first rebound of a ball when dropped from various heights. Tabulate and graph.

Initial height (cm)	Height rebound (cm)	$h \div x$
20		
40		
60		
.		
.		
.		
x		

1. Is the graph a straight line or a curve?

2. From your graph, what will be the height of the bounce if the initial height is:
(a) 35 cm? (b) 118 cm? (c) 190 cm?

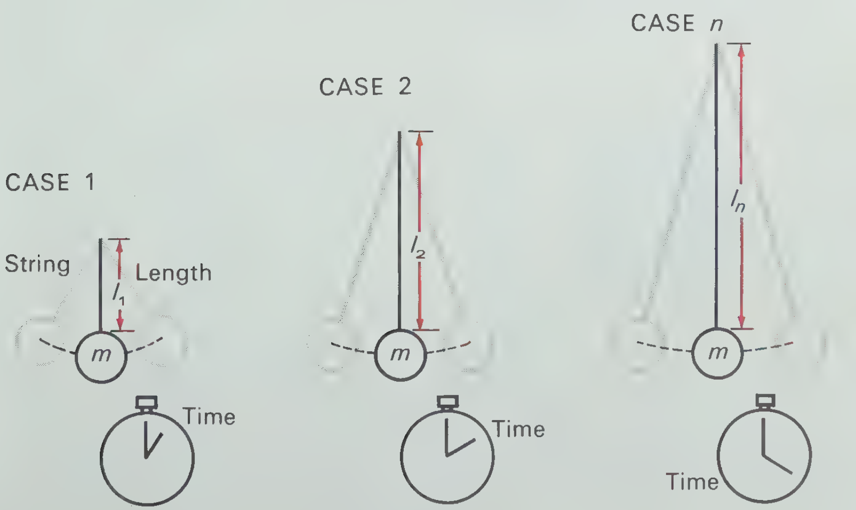
3. From what initial height must the ball be dropped in order to have it bounce:
(a) 47 cm? (b) 82 cm? (c) 136 cm?

4. From the table of values, divide the height of bounce by the initial height for each case and tabulate. What do you observe?

5. If the initial height is represented by x, and the bounce by h, write an algebraic equation to show their relation.

INVESTIGATION 1.8 *Pendulums*

Measure the period of a pendulum as the length of the string is increased. NOTE: the period is the time in seconds for one complete swing. The mass of the "bob" of the pendulum is constant throughout the investigation. Tabulate and graph length of pendulum against period.



Dow Corning Silicones
Inter-America Ltd.

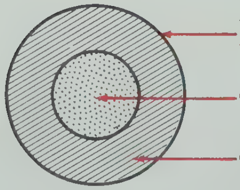
For maximum flight a golf ball must have maximum resilience, or bounce. The simplest method of determining resilience is to drop the ball from a given height and measure the height of its rebound. If a ball bounces to 7 m off a hard surface after being released from a height of 10 m, it has a resilience of 70% (the remaining 30% is the energy converted into heat).

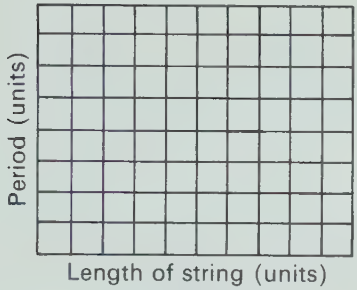
For maximum resilience golf balls are usually made with solid polybutadiene rubber core wound with a natural rubber thread stretched at 800 to 1000% elongation. The high tension in the rubber wrapping represents "stored" energy, which is released against the face of the golf club when hit. This release of energy is what determines how far the ball travels. Bounce is sometimes an undesirable quality, for instance when objects are parachuted from a plane. The ball on the left in the picture is made from Sylgard, a resin developed by Dow Corning Ltd., designed to "encapsulate" (absorb the bounce within itself) and thus protect electronic devices from shock when air-dropped.

When a golf ball is hit, the rapid increase of stress results in very little increase in strain. This maximizes the storage of energy which is released giving the ball the greatest distance. If a golf ball is wound under low tension then it requires a greater energy input while hitting the ball in order to achieve the same distance obtained with a ball wound under high tension, or it gives less distance for any given impact energy.

This tension is measured on finished golf balls by means of a compression tester. This tester applies a load of 550 kg to a ball, notes the amount of deformation of the ball and gives a reading of this deformation subtracted from a standard value. Thus the harder the ball, the less the deformation and the higher the compression reading. Top grade balls, with the greatest amount of thread wound at high tension, are designed for the highest compressions and give the greatest distance. Cheaper grades are designed to use less of the costly elastic thread and give somewhat reduced performance.

The general features of a golf ball are:





Case	Length l (units)	Period T (units)	$T \div l$
1			
2			
3			
.			
.			
.			
n			

Place 8 queens on a chessboard so that no queen attacks another queen.

1. Is the graph a straight line or a curve?

2. From your graph, find the period of the pendulum if the lengths are:
(a) 15 units (b) 25 units (c) 45 units

3. Similarly, from your graph find the length of your pendulum if you wanted a period of:
(a) 1 s (b) 1.5 s (c) 2 s

4. If the length of the pendulum is doubled, how is the time of the swing affected?

5. From the table of values, divide the period of the pendulum by the length and tabulate. What do you observe?

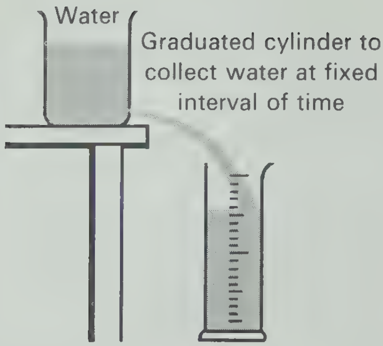
6. What is the algebraic equation which relates the length of the pendulum l and the period T?

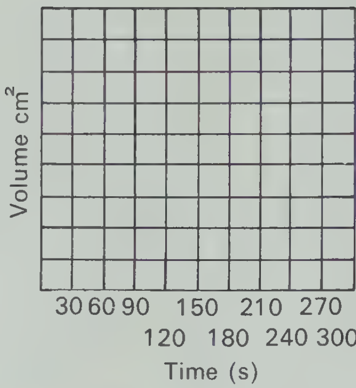
7. Why is a pendulum used in grandfather clocks?

INVESTIGATION 1.9 *Water Pressure*

Measure the amount of water collected in a fixed interval of time as the water flows from a small hole at the base of a tall container.

Tabulate and plot the time against the amount collected in the graduated cylinder. Draw the curve of best fit.





Time t (s)	Amount in cylinder V (cm^3)	$V \div t$
30		
60		
90		
120		
.		
.		
.		
n		

1. Is the graph a straight line or a curve?

2. Does the curve of best fit necessarily pass through all the points? Why?

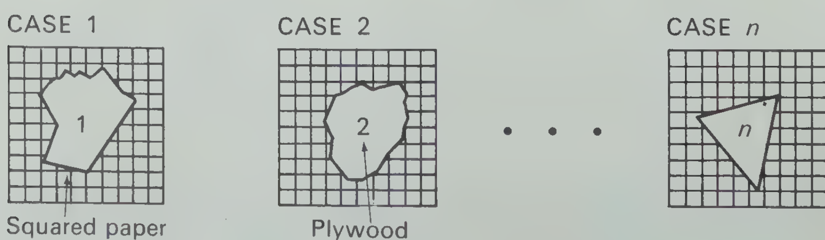
3. From your graph read the amount of water that should be in the graduated cylinder at:
(a) 135 s (b) 225 s (c) 320 s

4. From the table of values, divide the volume by the time for each case and tabulate. What do you observe?

5. Add an additional column to your table of values by multiplying the volume by the time in each case. What do you observe?

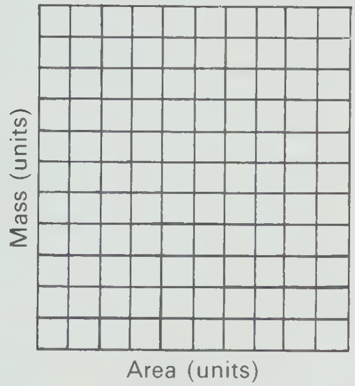
INVESTIGATION 1.10 *Irregular Objects*

Compare the area and the mass of irregularly shaped flat objects.



Place irregularly shaped pieces of plywood on squared paper and trace the outline. By counting squares, estimate the areas of each. With an accurate balance, find the mass of each object. Tabulate and graph area against mass.

Object	Area A (units)	Mass m (units)	$m \div A$
1			
2			
3			
.			
.			
.			
n			



1. Is the graph a straight line or a curve?

2. From the table of values, divide the mass by the area in each case and tabulate. What do you observe?

3. As the area of the plywood increases, how is the mass affected? How is this indicated on the graph?

4. If the plywood is doubled in thickness and the shape left the same, how will this affect the mass?

5. If all the irregular shapes are doubled in their thickness, draw another line on the graph to indicate the result.

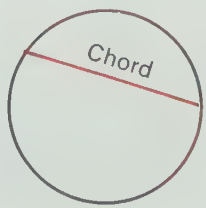
1.2 RELATIONS AND FUNCTIONS

In your investigations you were asked to compare one quantity with another to try to find a relationship between them. From the tables of values that you made in each of the investigations, you noted that for every change in the first quantity, there was a corresponding change in the second.

For example:

Diameter	5.0	3.5	7.0	2.5	4.0	1.8	2.0	14.0	10.0
Circumference	15.7	11.0	22.0	7.9	12.6	5.7	6.3	44.0	31.4

What is the largest number of regions into which three chords can divide a circle?



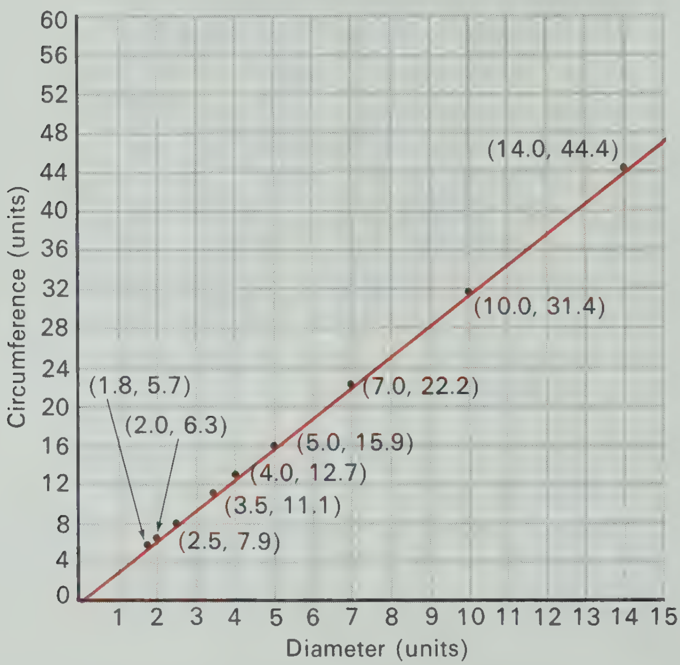
In each of the cases the results can be expressed as a pair of numbers in the form (d, c), where d represents the diameter and c the circumference. Therefore a diameter of 5.0 units and a circumference of 15.7 units can be written (5.0, 15.7). Note that in this notation the order of the numbers is very important. If the numbers were interchanged and the pair written (15.7, 5.0) this would imply a diameter of 15.7 units and a circumference of 5.0 units. Thus values written in the form (d, c) are generally referred to as ordered pairs.

The table of values can be written as a set of ordered pairs, e.g. {(5.0, 15.7), (3.5, 11.0), (7.0, 22.0), (2.5, 7.9),. . .}. Such a set of ordered pairs is referred to as a *relation*, as it indicates how one number is related to another.

A relation that has one, and only one, value for the second component for each value of the first component is called a *function*. All the relations in our investigations were functions.

EXAMPLE 1.

The graphical representation of an ordered pair is a point.



Since there are infinitely many ordered pairs possible, the complete set of ordered pairs will fall on a line as represented. Hence a line is a graphic representation of the complete set. If the line is a straight line, then the relation is said to be a linear relation.

EXERCISE 1-2

1. From the given tables of values, draw the graph, and from the graph, fill in the missing values in the tables. Reading such values from a graph is known as *interpolation*.

(a) Cost of fuel.

No. litres	2.0	3.0	5.0	8.0	4.3		9.2		1.0
Cost ($)	0.92	1.38	2.30	3.68		1.60		2.62	

(b) Distance travelled in a car at constant speed.

No. h	3	$4\frac{1}{2}$	$5\frac{1}{4}$	$\frac{3}{4}$	$1\frac{1}{2}$		$3\frac{1}{4}$		1
Dist. (km)	168	252	294	42		224		322	

(c) Property tax due to assessment.

Assessment ($)	5900	6800	7600	8600	6300		8100		1000
Tax ($)	281	324	362	410		286		376	

2. Canada's growth in population from 1959 to 1968 is indicated in the following table.

Year	1959	1960	1961	1962	1963	1964	1965	1966	1967	1968
Pop. (millions)	17.5	17.9	18.2	18.6	18.9	19.2	19.6	20.0	20.4	20.7

(a) Represent this data on a graph by drawing a curve of best fit.
(b) If the census figure in the table was for 06-01 in each case, from your graph give the approximate population of Canada on:
(i) 1964-01-01 (ii) 1967-04-01 (iii) 1961-10-01

3. The following table indicates the number of persons living at a particular age.

Age	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85
No. persons (thousands)	100	97	92	88	85	81	79	74	69	64	58	48	39	26	16	5

(a) Draw a graph to represent this data.
(b) By interpolating from your graph, approximate the number of persons alive at age:
(i) 17 (ii) 48 (iii) 69

4. The temperature reading for every hour was recorded on a given day in the table.

Time	06	07	08	09	10	11	12	13	14	15	16	17	18
Temp. °C	10	13	17	20	23	28	30	29	25	21	15	12	11

(a) Draw a graph to represent this data.
(b) From your graph, approximate the temperature at:
(i) 10:30 (ii) 14:30
(c) During which period of 2 h did the temperature rise most rapidly?

(d) During which period of 2 h was the temperature most nearly constant?

(e) When was the temperature falling most rapidly?

5. The electrical load (kilowatts) on a plant from 15:00 to 24:00 is indicated in the following table.

Time (h)	15	16	17	18	19	20	21	22	23	24
Load (kW)	50	60	76	120	140	150	142	100	45	30

What is the largest number of regions into which four chords can divide a circle?

(a) Represent the data on a graph by drawing the curve of best fit.

(b) What was the load at: (i) 18:30? (ii) 20:45?

(c) At what times was the load 70 kW?

(d) Account for the shape of the curve.

6. An engine is used for lifting masses. The values in the table indicate the masses lifted by the engine and the efficiency as a percent.

Mass (kg)	0	5	10	15	20	25	30	35	40
Efficiency (%)	0	58.3	70.2	75.6	78.1	80.0	81.3	81.5	81.8

(a) Graph the data.

(b) What is the efficiency when the mass on the engine is:

(i) 7 kg? (ii) 13 kg? (iii) 24 kg?

(c) If the engine is working at an efficiency of 61% what load is it lifting?

(d) For what range of mass does efficiency increase most rapidly?

(e) For what range of mass does efficiency increase very little?

1.3 DIRECT VARIATION

By examining the table of values in the relationship between the diameter and the circumference of a circle, we noted that when the circumference was divided by the diameter, the quotient was always equal to 3.141 59. . . .

Diameter d	5.0	3.5	7.0	2.5	4.0	1.8	2.0	14.0	10.0
Circumference c	15.9	11.1	22.2	7.9	12.7	5.7	6.3	44.4	31.4
$c \div d$	3.14	3.14	3.14	3.14	3.14	3.14	3.14	3.14	3.14

The Greeks long ago discovered this fact and gave the value 3.14 . . . the special letter π. The relationship between circumference and diameter was written in the form of an algebraic equation $c = \pi d$. We have seen from our investigation that the graph of the relation $c = \pi d$ is a straight line. When two variables x and y are related such that $\frac{y}{x} = k$, a constant, we say that y varies directly as x. If $\frac{y}{x} = k$, it follows that $y = kx$.

Here are a few examples of direct variation.

(a) If a 50 g mass on a spring produces a stretch of 10 cm, and if the mass is then increased to 100 g, the stretch will be 20 cm. (Stretch varies directly with the mass.)

(b) If a ball is dropped 12 cm, and the resulting height of bounce is 8 cm, and if the ball is then dropped 36 cm, the height of bounce will be 24 cm. (Height of bounce varies directly with the initial height.)

(c) If a car travels 60 km in 1 h, then the car will travel 120 km in 2 h. (Distance varies directly as time, speed being constant.)

Test your skill: 0.007×0.03

EXAMPLE 1. *In an electrical circuit the voltage varies directly with the current. If the power is 9 V when the current is 6 A, find an equation representing the relation between voltage and current.*

Solution Since the voltage varies directly as the current, we know that the ratio voltage (V) to current (I), is always the same.

Therefore $\frac{V}{I} = k$, where k is a constant. To find k, we know that $V = 9$ and $I = 6$ A. Therefore $\frac{9}{6} = k$ or the constant $k = 1.5$. The relation between V and I expressed as an equation is $\frac{V}{I} = 1.5$ or $V = 1.5I$.

Note: This equation $V = 1.5I$ is of the form $y = kx$, where y corresponds to V and x corresponds to I.

EXERCISE 1-3

1. The length of stretch in a spring will vary directly as the mass attached to it. If the spring is stretched 3.4 cm by a mass of 250 g, what will the stretch be if 360 g are attached to the spring?

2. The expense of building a sidewalk varies directly as its length. If the cost of 2000 m is $30 000, how much will a sidewalk 3200 m long cost?

3. The distance d in kilometres which a jet travels at uniform speed varies directly as the elapsed time t in hours. If it covered 575 km in 2.5 h:

(a) Find the distance the jet would travel in 4 h.

(b) Find the time it would take to travel 690 km.

4. The acceleration of an object varies directly with the force applied to it. An acceleration of 62 m/s requires a force of 4 kN. What force is required to produce an acceleration of 93 m/s?

5. The interest on a loan varies directly with the number of days of the loan. If the interest for 1 a is $48.30, how much interest should be paid from 03-23 to 11-30 inclusive?

a is a short form of annum, the Latin word for year.

1.4 PARTIAL VARIATION

EXAMPLE 1. *A club wants to buy distinctive greeting cards for a fund-raising project. Publisher A will sell them the cards at 20¢ per card. Publisher B makes an initial charge of $50, plus 10¢ per card. If the*

club expects to sell (a) 1000 cards, (b) 300 cards, which plan is better? How many cards would there have to be in order that both costs would be the same? Draw a graph to illustrate the two plans.

Solution

Plan A

Cost = (Cost per card) × (Number)

$C = \$0.20N$

(a) $C = \$0.20 \times 1000 = \$200$

(b) $C = \$0.20 \times 300 = \$60$

Plan B

Cost = Initial cost + (Cost per card × Number)

$C = \$50 + \$0.10N$

(a) $C = \$50 + \$0.10(1000) = \$50 + \$100 = \$150$

(b) $C = \$50 + \$0.10(300) = \$50 + \$30 = \$80$

Therefore it is better to buy from Publisher B for the 1000 cards and from Publisher A for the 300 cards.

Plan A

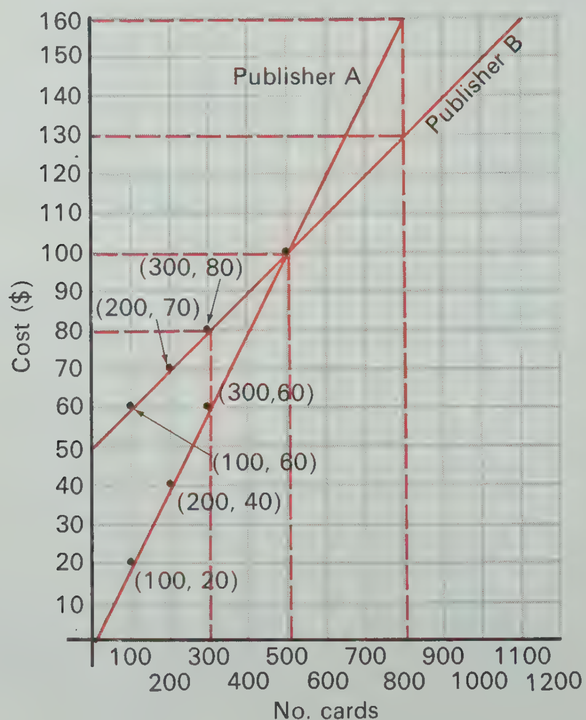
$C = 0.20N$

No.	100	200	300
Cost ($)	20	40	60

Plan B

$C = 50 + 0.10N$

No.	100	200	300
Cost ($)	60	70	80



From the graph we see that the point common to both plans corresponds to the purchase of 500 cards. At 500 either plan will cost the same. If more than 500 cards are sold, cards from Publisher B will be more economical. If less than 500 cards are sold, cards from Publisher A will cost less. The graph shows the saving in each case.

Addition:
EAT
MEAT
AS
A
SNACK

EXERCISE 1-4

1. E-Z-Rent-a-Car Agency charges \$5/d plus 8¢/km for one of their cars. When a car is picked up at 18:00 on Friday night the odometer reads 21 837 km. When the car is returned on Sunday at 17:00 the odometer reads 22 161. How much does the weekend's drive cost?

2. TV Care charges \$4.50 for a service call and \$2.95 for each of the three tubes needing replacement. How much is the repair bill?

3. It costs \$16/h to rent a bulldozer. Moving the bulldozer to a job site and taking it away after the job is finished cost a flat rate of \$35. If you have a job requiring 12 h work, how much will the bulldozer cost?

4. Many salesmen work on a certain basic salary plus commission. Mr. Cell receives a basic salary of \$75 a week plus 5% of sales. Last week he made the following sales: Monday \$210.35, Tuesday \$226.65, Wednesday \$112,60, Thursday \$230.50, Friday \$350.30 and Saturday \$246.50. What was his salary for last week?

5. Publisher A will print posters for an initial charge of \$15 plus 10¢ a poster. Publisher B has an initial charge of \$10 plus 15¢ a poster.
(a) What publisher gives the better price for 50 posters and what is it?
(b) For what number of posters is the price the same?

6. To rent an automobile Rent-a-Car charges \$6/d plus 10¢/km. Lend-a-Car charges \$10/d with 50 km/d included and 12¢/km after that.
(a) Graph the cost of renting a car from each company for 0 to 300 km for one day.
(b) What will each company charge for renting a car for one day and driving 100 km?
(c) What is the cheaper price for renting a car for one day and driving 200 km?
(d) What is the cheaper price for two days and a total distance of 200 km?

REVIEW AND PREVIEW TO CHAPTER 2

EXERCISE 1 *Operations with Decimals*

1. Add

(a) $\begin{array}{r} 0.176 \\ 3.008 \\ \underline{0.004} \end{array}$ (b) $\begin{array}{r} 4.003 \\ \underline{0.147} \end{array}$ (c) $6.007 + 0.013 + 4.2$

(d) $6.7 + 0.018 + 13.2$ (e) $4.6 + 0.572 + 4.0$

2. Subtract

(a) $\begin{array}{r} 4.007 \\ \underline{0.083} \end{array}$ (b) $\begin{array}{r} 7.362 \\ \underline{1.789} \end{array}$ (c) $0.571 - 0.073$

(d) $1.000 - 0.476$ (e) $2.002 - 0.983$

3. Multiply

(a) 0.073×4.6 (b) 0.007×5.0 (c) 9.001×0.2

(d) $\begin{array}{r} 8.63 \\ \underline{0.07} \end{array}$ (e) $(0.58)(4.623)$

4. Divide

(a) $0.3358 \div 0.073$ (b) $\frac{60.19}{0.13}$ (c) $\frac{2.2022}{1.1}$

(d) $0.9348 \div 1.23$ (e) $4.224 \div 0.6$

EXERCISE 2 *Exponents*

Simplify

1. $(a^4)(a^7)$
2. $(b^3)(b^5)(b^7)$
3. $(2m^4)(4m^6)$
4. $(-3xy)(-2x^2y)$
5. $(4mn)(3m^2)(-2n)$
6. $(6x^3y)(-3xy^4)$
7. $(-2ab^2)(-4ab)(2b^2)$
8. $a^4 \div a^2$
9. $x^7 \div x^5$
10. $(8m^2) \div (4m)$
11. $(-12a^2b^2)(-3ab^2)$
12. $(20a^4b^6) \div (-5a^3b^4)$
13. $(-60a^3x^6) \div (-12a^3x^6)$
14. $(-3a^2b)(-4b^2) \div (6ab)$

EXERCISE 3 *Evaluating Polynomials*

If $a = 2$, $b = -3$, $c = 4$ and $d = -1$, evaluate the following.

1. $3ac + a^2$
2. $2ab - 3bc$
3. $2d - 3bc + a$
4. $a^2 + b^2$

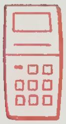
5. $4a^2 - 3d$ 6. $2b^2d - 3d^2$ 7. $\dfrac{4abc}{d}$ 8. $6d^2 - 3b^2c$

9. $\dfrac{-3ab - 2c}{-5}$ 10. $a^2 + c^2 - d^2$ 11. $-4a^2 - 2c^2 + d$

12. $12abcd$ 13. $21d - 3a^2b^2c^2$ 14. $\dfrac{6ac - 4d^2}{-2d}$

15. $a^2b^2 - c^2d^2$ 16. $3cd - 1$ 17. $5 + 6abc^2$

Perform the following calculations:



1. $\dfrac{3.215 + 4.275}{6.375 - 4.635}$

2. $\dfrac{6.735 \times 21.26}{4.871 + 86.35}$

3. $\dfrac{0.3755 - 0.2175}{1.275 \times 6.315}$

4. $\dfrac{5.255 \times 8.375}{2.125 \times 4.615}$

5. $\dfrac{63.25 - 57.84}{21.64 \div 3.215}$

6. $\dfrac{21.85 \div 4.675}{53.55 - 42.65}$

7. $\dfrac{0.5345 - 0.6745}{25.37 - 31.25}$

8. $\dfrac{6.485 \div 21.25}{6.785 \times 3.125}$

9. $\dfrac{5.845 \times 8.614}{28.37 + 41.25}$

10. $\dfrac{3.125 - 8.675}{6.725 + 8.125}$

CHAPTER 2

Algebra

In order to analyze the relationships between quantities, mathematicians have developed a branch of mathematics called algebra. The skills developed in studying algebra are used in the practical branches of mathematics.

2.1 PRODUCT OF A MONOMIAL AND A POLYNOMIAL

The distributive property is used when multiplying a polynomial by a monomial. For example:

$a(b + c) = ab + ac$

$$\begin{aligned} 2(3x - 2) &= 2(3x - 2) \\ &= 2(3x) + 2(-2) \\ &= 6x - 4 \end{aligned}$$

By recalling the "rules of signs for multiplication" we can expand and simplify algebraic expressions.

$(+3) \times (+2) = (+6)$
$(-3) \times (+2) = (-6)$
$(+3) \times (-2) = (-6)$
$(-3) \times (-2) = (+6)$

EXAMPLE 1. *Expand the following and simplify*
(a) $2(x + 3) + 3(x - 4)$
(b) $-3(2a - 4) - (3a - 2)$
(c) $2x(x - 4) - x(2x + 3)$

Solutions

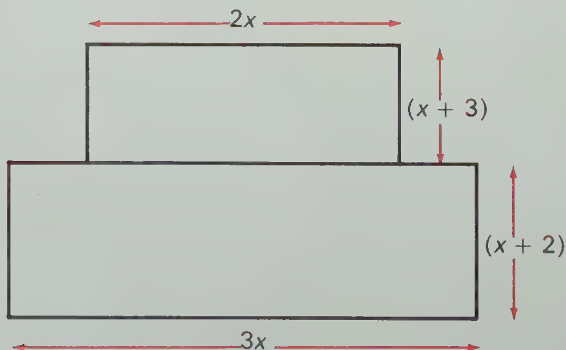
(a) $$\begin{aligned} 2(x + 3) + 3(x - 4) &= 2(x + 3) + 3(x - 4) \\ &= 2x + 6 + 3x - 12 \\ &= 5x - 6 \end{aligned}$$

The unwritten factor in front of a bracket is always "1" or "-1".

(b) $$\begin{aligned} -3(2a - 4) - (3a - 2) &= -3(2a - 4) - 1(3a - 2) \\ &= -6a + 12 - 3a + 2 \\ &= -9a + 14 \end{aligned}$$

(c) $$\begin{aligned} 2x(x - 4) - x(2x + 3) &= 2x(x - 4) - x(2x + 3) \\ &= 2x^2 - 8x - 2x^2 - 3x \\ &= -11x \end{aligned}$$

EXAMPLE 2. *Find the area of the given figure.*



Solution The area is

$$\begin{aligned} &2x(x + 3) + 3x(x + 2) \\ &= 2x^2 + 6x + 3x^2 + 6x \\ &= 5x^2 + 12x \end{aligned}$$

EXERCISE 2-1

A Expand each of the following.

1. (a) $2(3x + 1)$ (b) $4(2a - 3)$ (c) $5(2a + 3b)$
(d) $-2(x - 7)$ (e) $-3(x - y)$ (f) $-(2x - 4)$
(g) $7(-3y - 2)$ (h) $2x(x + 2)$ (i) $-2x(x - y)$

2. (a) $3(x - y - t)$ (b) $-4(2a - 3b - c)$
(c) $-(2p - 3q - 6)$ (d) $2a(x - y - t)$
(e) $2x(x^2 - 3x - 3)$ (f) $ab(a^2 - b - 1)$

B Expand and simplify.

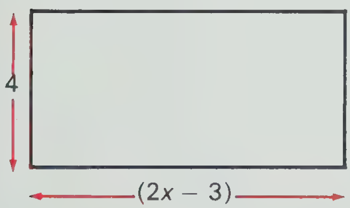
3. (a) $2(a + 3) + 3(a - 2)$ (b) $4(y - 3) - 2(y - 7)$
(c) $3(x + 3) - (x - 7)$ (d) $2(4b + 1) - 3(1 - 2b)$
(e) $4(a - 1) - (a - 3) - 2(a + 7)$
(f) $-3(1 - 2x) + (2x - 1) - (3x - 2)$
(g) $4(x^2 - x - 2) - 3(2x^2 - x + 1)$
(h) $2(a - 3b + c) - (a - 4b - 4c)$
(i) $-3(a^2 - 2a - 1) + 7(1 - 2a)$
(j) $3x - 2(x - 7) - 3(1 - 2x^2)$

What is the largest number of regions into which five chords can divide a circle?

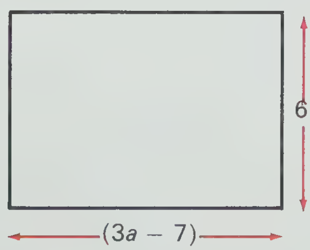
4. (a) $x(2x - 1) - 2x(x + 3)$ (b) $4a(a - b) - 3a(b - a)$
(c) $2b(1 - 3b) + 4(2b - 1)$ (d) $-3m(m - n) - 2n(m - n)$
(e) $-4(x - 3xy) + x(x - y)$ (f) $2a(3a - 1) + 2a^2 - (a - 7)$
(g) $2(x^2 - 3x + 1) - x(x - 4)$ (h) $-2t(1 - 3t) - (2t^2 - 3t + 2)$
(i) $7 - 2(a - 3b + 4) - 4(a + b)$ (j) $6t^2 - 3(2t^2 - t - 1) + 7t$

5. Find the area of the following.

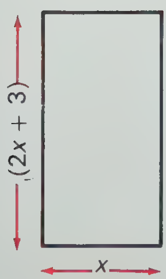
(a)



(b)



(c)



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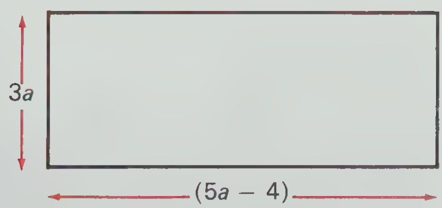

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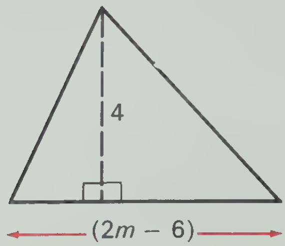
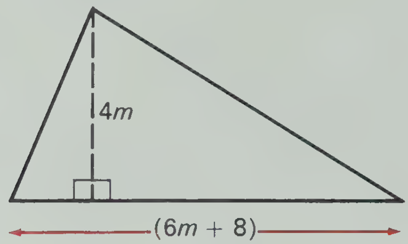
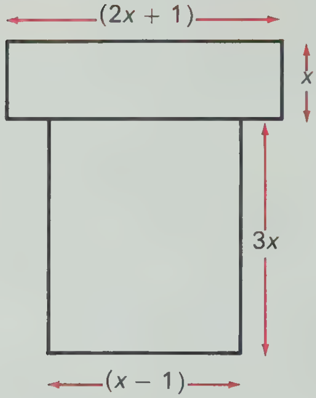


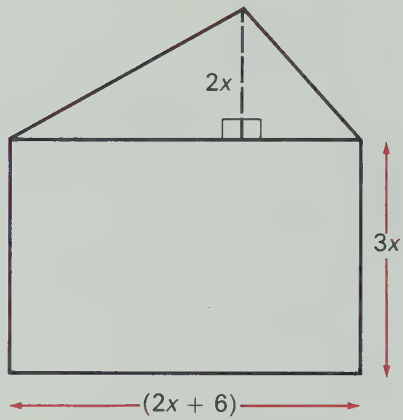
(f)



(g)



(h)



C Expand and simplify.

Remove "inside" brackets first.

6. (a) $[3m - 2(m - 3)] + 4[7 + 3(m + 2)]$
(b) $-2[3x - 4(x + 1)] - 7(x + 3)$
(c) $3[2(1 - a) - 3a] - [4a - 2(a + 1)]$
(d) $7 - 3[2b + 4(1 - 3b)] + [4 - 3(2b - 1)]$
(e) $3b^2 + 2b[7 - 3(b + 1)] - 5[2b(1 - 3b) - 2]$

7. (a) $xy(x - y) - 3(x^2y + x)$
(b) $a^2b(b - a) - a^2b(b + a)$
(c) $2mn(3m - n) - 4mn(2m + 3n)$
(d) $\frac{1}{2}(2x - 4y) - \frac{1}{4}(8x + 12y)$
(e) $\frac{2}{3}(6y - 9b) + \frac{4}{3}(3b + 12y) - \frac{1}{3}(-6b - 3y)$

2.2 PRODUCT OF TWO BINOMIALS

The distributive property is used to expand the product of two binomials.

EXAMPLE 1. *Expand and simplify (x + 2) (x + 3)*

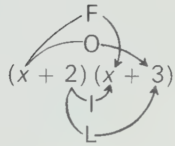
Solution

$$\begin{aligned}(x + 2)(x + 3) &= (x + 2)(x + 3)\\ &= (x + 2)x + (x + 2)3\\ &= x^2 + 2x + 3x + 6\\ &= x^2 + 5x + 6\end{aligned}$$

The same result is obtained using the following procedure

$$(x + 2)(x + 3) = (x + 2)(x + 3)$$
$$= x^2 + 3x + 2x + 6$$
$$= x^2 + 5x + 6$$

Each term in the first binomial multiplies each term in the second binomial.



EXAMPLE 2. *Expand and simplify $(2x - 3)^2$*

Solution

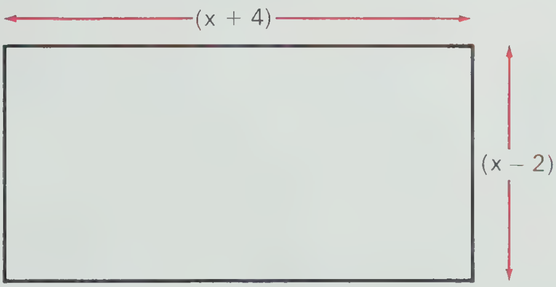
$$(2x - 3)^2 = (2x - 3)(2x - 3)$$
$$= (2x - 3)(2x - 3)$$
$$= 4x^2 - 6x - 6x + 9$$
$$= 4x^2 - 12x + 9$$

EXAMPLE 3. *Expand and simplify $(2x - y)(2x + y)$.*

Solution

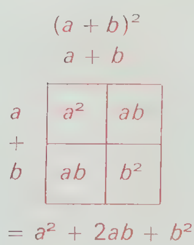
$$(2x - y)(2x + y) = (2x - y)(2x + y)$$
$$= 4x^2 + 2xy - 2xy - y^2$$
$$= 4x^2 - y^2$$

EXAMPLE 4. *Find the area of the given figure.*



Solution The area is

$$(x + 4)(x - 2)$$
$$= x^2 - 2x + 4x - 8$$
$$= x^2 + 2x - 8$$



EXERCISE 2-2

B Expand the following and simplify.

1. (a) $(x + 2)(x + 3)$ (b) $(a - 3)(a + 7)$ (c) $(b - 7)(b - 2)$
(d) $(y - 2)(y - 3)$ (e) $(m - 1)(m + 1)$ (f) $(t + 2)(t + 3)$
(g) $(c - 2)(c - 6)$ (h) $(x - 5)(x - 4)$ (i) $(m + 7)(m + 6)$
(j) $(2x - 1)(x + 3)$ (k) $(3b + 2)(2b - 1)$ (l) $(5a - 1)(5a + 1)$

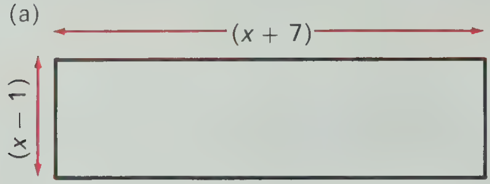
2. (a) $(x + 7)^2$ (b) $(m - 2)^2$ (c) $(a + 3)^2$

(d) $(2x + 1)^2$ (e) $(4y - 3)^2$ (f) $(1 + 3x)^2$
(g) $(2m + 5)^2$ (h) $(1 - 5x)^2$ (i) $(2r - 7)^2$

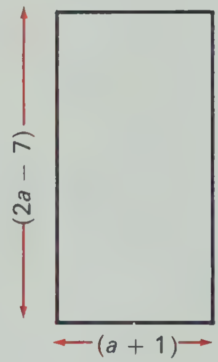
3. (a) $(4a - 3x)(2a + x)$ (b) $(x - 2y)(x + 3y)$
(c) $(2a - b)(a + b)$ (d) $(3m - 2b)(3m + 2b)$
(e) $(x - 2y)^2$ (f) $(2a + b)^2$
(g) $(2d - 3g)(7d + 3g)$ (h) $(2x - y)(2x + y)$
(i) $(6a - 7b)^2$ (j) $(2m - 1)(m + 4)$
(k) $(1 - 3x)(2x + 5)$ (l) $(4 - 3x)(5 - 7x)$

4. Find the area of the following figures.

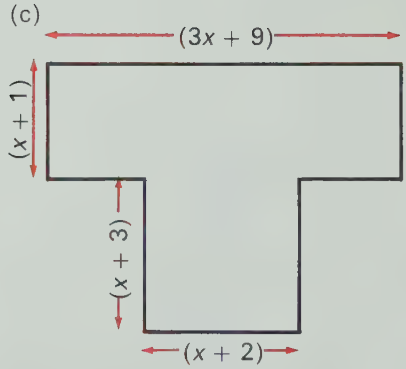
(a)



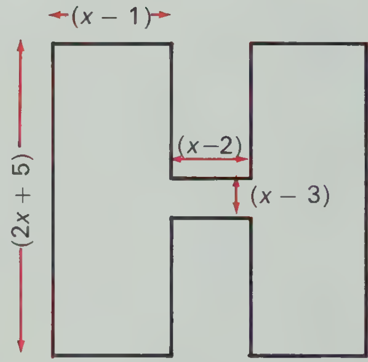
(b)



(c)



(d)



EXAMPLE 5. *Expand and simplify $(x + 3)(x^2 + 2x - 1)$.*

Each term in the binomial multiplies each term in the trinomial.

Solution

$$(x + 3)(x^2 + 2x - 1) = (x + 3)(x^2 + 2x - 1)$$
$$= x^3 + 2x^2 - x + 3x^2 + 6x - 3$$
$$= x^3 + 5x^2 + 5x - 3$$

C Expand and simplify.

5. (a) $(x + 1)(x^2 + 2x + 1)$ (b) $(a + 2)(a^2 + 3a - 1)$
(c) $(b + 1)(b^2 - b - 1)$ (d) $(y - 4)(y^2 + 3y + 2)$
(e) $(x + 2)(2x^2 - x + 1)$ (f) $(m^2 + 2m - 1)(m + 1)$
(g) $(x^2 + x + 1)(x^2 - x - 1)$ (h) $(2x^2 - x + 3)(x^2 + 2x - 1)$
(i) $(a^2 + 3a + 4)(a^2 - 5a - 7)$ (j) $(2y^2 - 3y - 6)(3y^2 + y + 2)$

2.3 EXPRESSIONS INVOLVING POLYNOMIALS

In this section the rules for multiplication of polynomials are applied to the simplification of expressions.

What is the largest number of regions into which six chords can divide a circle?

EXAMPLE 1. *Simplify* $(2x - 1)(3x + 2) - 4(x - 7)$.

Solution

$$\begin{aligned} &(2x - 1)(3x + 2) - 4(x - 7) \\ &= (6x^2 + 4x - 3x - 2) - 4x + 28 \\ &= (6x^2 + x - 2) - 4x + 28 \\ &= 6x^2 + x - 2 - 4x + 28 \\ &= 6x^2 - 3x + 26 \end{aligned}$$

EXAMPLE 2. *Simplify* $3(a - 2)(2a - 1) - 2(a + 3)(3a + 2)$.

Solution

$$\begin{aligned} &3(a - 2)(2a - 1) - 2(a + 3)(3a + 2) \\ &= 3(2a^2 - a - 4a + 2) - 2(3a^2 + 2a + 9a + 6) \\ &= 3(2a^2 - 5a + 2) - 2(3a^2 + 11a + 6) \\ &= 6a^2 - 15a + 6 - 6a^2 - 22a - 12 \\ &= -37a - 6 \end{aligned}$$

EXERCISE 2-3

B Simplify the following.

Test your skill: $0.0015 \div 0.03$

1. (a) $2(x - 3) + (x + 1)(x + 4)$ (b) $2(m - 3) + (m + 2)(m - 3)$
(c) $(4x - 1)(x + 3) + (2x - 3)(x - 1)$
(d) $(4a + 1)(4a - 1) + (a + 3)(a + 2)$

2. (a) $2(2x - 1) + 3(x + 2)(x - 4)$
(b) $3(a - 2)(a - 3) + 2(a + 6)$
(c) $2(m - 1)(m + 3) + 4(m - 2) + 6$
(d) $3(2x + 1)(x + 2) + 2(x - 1)(2x + 3)$
(e) $(x + 3)^2 + (x - 2)^2$ (f) $2(a - 3)^2 - (a + 1)^2$

3. (a) $2(a - 2)(a + 2) - 3(a + 4)(a + 6)$
(b) $2(t - 3)(t + 4) - (3t + 1)(t - 6)$
(c) $5(m + 3)(2m - 1) - (m + 2)$
(d) $3(a + 6)(2a + 1) - 4(2a - 1)(a + 3)$
(e) $2(x + 1)^2 - (x + 2)(x + 3)$
(f) $(m - 1)^2 - 3(2m - 1)^2$

4. (a) $-3(2m - 1)(m + 2) - 4 + 3m$
(b) $3(1 - 2x)(1 + x) - 2(3x + 7)(x - 4)$
(c) $3(x - 3)(7 - x) - 5(2x - 1)(2x + 1)$
(d) $-4(t + 3)(2t - 1) - 6(1 - 3t)$
(e) $3(m + 1)(m + 2) - (2m - 1)^2$

C **5.** (a) $(x + 3)(x^2 + 2x - 1) + (2x + 3)(x + 2)$
(b) $(a^2 - 2a + 1)(a - 3) - (a^2 + 3a + 2)(a + 1)$
(c) $(y^2 + 2y + 1)(2y + 1) + 2(y^2 - 3y - 2)(y - 4)$
(d) $(2a^2 + a - 4)(3a - 1) - 3(2a^2 - 2a - 1)(3a + 1)$

2.4 DIVISION OF A POLYNOMIAL BY A MONOMIAL

The division of two monomials is illustrated in the examples.

$$\frac{12ab}{3b} = \frac{12}{3} \cdot a \cdot \frac{b}{b} = 4a$$

EXAMPLE 1. *Simplify* $\dfrac{12ab}{3b}$

Solution

$$\frac{12ab}{3b} = \boxed{\frac{(4a)(3b)}{(3b)}}$$
$$= 4a$$

EXAMPLE 2. *Simplify* $\dfrac{15ax + 3ab}{3a}$

Solution

$$\frac{15ax + 3ab}{3a} = \boxed{\frac{15ax}{3a} + \frac{3ab}{3a}}$$
$$= 5x + b$$

EXERCISE 2-4

A Simplify

1. (a) $\dfrac{10ab}{2a}$ (b) $\dfrac{24bc}{6c}$ (c) $\dfrac{-15xy}{3x}$

(d) $\dfrac{-21abc}{-7ab}$ (e) $\dfrac{30dc}{-6dc}$ (f) $\dfrac{24a^2}{12a}$

(g) $\dfrac{-36a^2b^2}{6a^2}$ (h) $\dfrac{-32a^2b}{-4ab}$ (i) $\dfrac{-30a^2b^2}{15a^2b}$

B Simplify

2. (a) $\dfrac{3x + 3m}{3}$ (b) $\dfrac{12a - 4b}{4}$ (c) $\dfrac{3a^2 - 3}{3}$

(d) $\dfrac{24ab - 6b}{-6}$ (e) $\dfrac{21mt - 7m}{7}$ (f) $\dfrac{2ab - 4a + 6}{2}$

3. (a) $\dfrac{12ab - 6ac}{3a}$ (b) $\dfrac{10ac - 2a}{2a}$ (c) $\dfrac{2x^2 - x}{x}$

(d) $\dfrac{16x^3 + 4x^2}{-2x^2}$ (e) $\dfrac{9ab - 6a^2b + 3ab^2}{3ab}$

(f) $\dfrac{-6mn - 3m - 9mt}{-3m}$

4. (a) $\dfrac{100a^2b - 50ab^2}{-10ab}$ (b) $\dfrac{-12x^4 - 8x^3 + 2x^2}{2x^2}$

(c) $\dfrac{10a - 20a^2 + 30a^3}{-10a}$ (d) $\dfrac{-4m^4n^2 + 2m^2n^4}{2m^2n}$

(e) $\dfrac{24a^3b^2 - 12a^2b^3 + a^2b^2}{-ab^2}$ (f) $\dfrac{-4ab + 16a^3b^2 - 12a^2b^3}{-4ab}$

Addition:
HAM
EGGS
MASH

C Simplify

5. (a) $\frac{12x^2}{x} + \frac{4xy}{y}$ (b) $\frac{12ab}{3a} + \frac{16b^2}{4b}$ (c) $\frac{15ab}{3b} - \frac{9ac}{3c}$

(d) $\frac{20a^2}{5a} - \frac{4ab}{-2b}$ (e) $\frac{12xy}{4y} - \frac{3x^2}{-x} + \frac{2x}{2}$

(f) $\frac{24a^2b}{6a} + \frac{15ab}{5} - \frac{7ab^2}{-b}$

6. Simplify

(a) $\frac{3ab^2 + 6a^2b}{3ab} + \frac{2a^2 - 2ab}{2a}$

(b) $\frac{20x^2y - 25xy^2}{5xy} + \frac{36x^2y - 30xy^2}{6xy}$

2.5 DIVISION OF A POLYNOMIAL BY A BINOMIAL

The method used to divide a polynomial by a binomial is similar to the method of long division in arithmetic.

EXAMPLE 1. *Divide $(x^2 - x - 12)$ by $(x - 4)$.*

Solution

$$\begin{array}{r|l} & x + 3 \\ x - 4 & \overline{x^2 - x - 12} \\ & \underline{x^2 - 4x} \\ & 3x - 12 \\ & \underline{3x - 12} \\ & 0 \quad \text{(Remainder)} \end{array}$$

$\therefore (x^2 - x - 12) \div (x - 4) = (x + 3)$.
Since we are dividing by $(x - 4)$ we place the restriction $x \neq 4$ on the variable.

EXAMPLE 2. *Divide $(5b + 12b^3 - 12b^2 + 1)$ by $(2b - 1)$.*

Solution Arrange both the divisor and dividend in descending powers of b.

$$\begin{array}{r|l} & 6b^2 - 3b + 1 \\ 2b - 1 & \overline{12b^3 - 12b^2 + 5b + 1} \\ & \underline{12b^3 - 6b^2} \\ & -6b^2 + 5b \\ & \underline{-6b^2 + 3b} \\ & 2b + 1 \\ & \underline{2b - 1} \\ & 2 \quad \text{(Remainder)} \end{array}$$

What is the restriction on "b" in this problem?

EXERCISE 2-5

A Arrange the following in descending powers of the variable.

1. (a) $2x^2 + 5x^3 - 3 + 4x$ (b) $5a + 6a^3 - 3a^2 + 7a^4$
(c) $1 - 4b - 7b^2$ (d) $7m^2 + 8m^4 - 3m + 1$
(e) $4t^4 - 3t^2 - 6 - 5t$ (f) $-4 + 3x^2 - 2x + 6x^4 - 7x^6$

B Divide the following.

2. (a) $(x^2 + 8x + 15) \div (x + 3)$ (b) $(a^2 - 7a - 12) \div (a - 3)$
(c) $(m^2 - 2m - 24) \div (m - 6)$ (d) $\dfrac{x^2 - 3x - 9}{x - 5}$

3. (a) $(1 - x + x^3 - x^2) \div (x + 1)$
(b) $(6a^2 + a^3 + 12 + 6a) \div (a + 4)$
(c) $(2b^3 + 3b^2 - 3b + 12) \div (b + 3)$
(d) $(4c^2 + 12c^3 - 7c + 2) \div (2c - 1)$
(e) $(8a^2 - 2 + 4a) \div (2a - 7)$
(f) $(-4y^2 + 6y^4 - 5y + 8 + 7y^3) \div (2y + 1)$

EXAMPLE 3. *Divide $(b^3 + 6b - 20)$ by $(b - 2)$.*

Solution

$0b^2$ is a "placeholder"

$$
\begin{array}{r}
b^2 + 2b + 10 \\
b - 2 \,\overline{)\, b^3 + 0b^2 + 6b - 20} \\
\underline{b^3 - 2b^2} \\
2b^2 + 6b \\
\underline{2b^2 - 4b} \\
10b - 20 \\
\underline{10b - 20} \\
0
\end{array}
$$

4. (a) $(a^2 - 16) \div (a - 4)$ (b) $(2x^3 - 12x - 18) \div (x - 3)$
(c) $(r^3 - 1) \div (r - 1)$ (d) $(5 + 12m^4 - m^2 - 5m) \div (2m + 1)$

C

5. (a) $(2x^5 + x^3 - 5x^2 + 2) \div (x - 1)$
(b) $(2y^3 + 10 + 15y^4 - 16y - 39y^2) \div (4y - 2 + 5y^2)$
(c) $(x^4 - x^3 + 7x + 5) \div (x^2 - 3x + 5)$
(d) $(2x^3 - 7x^2y + 7xy^2 - 2y^3) \div (x - 2y)$

6. The area of a rectangle is $3x^4 - 4x^3 - 19x^2 - 7x + 2$. If the width is $3x + 2$, find the length.

7. If $R = \dfrac{E}{I}$, find R when $E = 6x^3 - 2x^2 - 9x + 3$ and $I = 3x - 1$.

2.6 LINEAR EQUATIONS PART 1

The skills developed in solving equations are essential to working with formulas, which appear often in science and technology. The solution set of an equation is the set of values for the variable which makes the left side equal to the right side.

EXAMPLE 1. *Solve and check $\{x \in R / 2(x - 4) - 3(x + 1) = 5 - 3x\}$*

Solution

$$
\begin{aligned}
2(x-4)-3(x+1) &= 5-3x\\
2x-8-3x-3 &= 5-3x\\
-x-11 &= 5-3x\\
2x &= 16\\
x &= 8
\end{aligned}
$$

Remove brackets, collect like terms.

Check: L.S. $= 2(x-4)-3(x+1)$
$= 2(8-4)-3(8+1)$
$= 2(4)-3(9)$
$= 8-27$
$= -19$

R.S. $= 5-3x$
$= 5-3(8)$
$= 5-24$
$= -19$

L.S. means "Left Side".
R.S. means "Right Side".

$\therefore \{8\}$ is the solution set of the equation.

EXAMPLE 2. *Solve for x.* $2(2x-1)-3(x-2)=3x+3$.

Solution

$$
\begin{aligned}
2(2x-1)-3(x-2) &= 3x+3\\
4x-2-3x+6 &= 3x+3\\
x+4 &= 3x+3\\
-2x &= -1\\
x &= \tfrac{1}{2}
\end{aligned}
$$

$\therefore$ The root is $\frac{1}{2}$.

EXERCISE 2-6

A

1. Solve the following equations.

(a) $x-7=10$ (b) $a+3=13$ (c) $m-4=12$
(d) $a-2=-8$ (e) $b+1=-11$ (f) $2+x=-6$
(g) $3x=15$ (h) $2x=40$ (i) $5a=-20$
(j) $-3a=-9$ (k) $-4m=16$ (l) $3b=0$

2. (a) $2x+1=7$ (b) $4b-2=10$ (c) $3a-3=0$
(d) $2+5m=22$ (e) $7=1+2x$ (f) $2x=x+3$
(g) $7r+5=6r+6$ (h) $2m+3=3$ (i) $1+3t=7$

3. Determine whether the given values are roots of the equation.

(a) $2x+1=7$; 3 (b) $3x-2=10$; 5
(c) $1+3x=10$; 3 (d) $4m-1=15$; 4
(e) $3y+2=4y$; 1 (f) $-2a=-4$; -1
(g) $3b+2=b$; -1 (h) $3x+2=2x+5$; 4

B

Solve and check.

4. (a) $4x-7=2x+3$ (b) $10y-6=4y+6$
(c) $3x+2-5x=-x+3$ (d) $4a-3=6a-11$
(e) $2+3y-4=-5y+6$ (f) $3a+6-7a=12-8a+2$
(g) $x+5=20-4x$ (h) $11x+2=26+7x$
(i) $-4c+4+2c=3c-16$ (j) $3-6b+6=2b-7$

5. (a) $5a+3(8-a)=36$ (b) $2k-5-(k-3)=7$
(c) $6x+1-3=12x-(8x-4)$ (d) $y-5=8-(2y+4)$
(e) $5(3x-4)=-14-3(x-10)$
(f) $6(x+1)-3=12x-4(2x-1)$

Starting with the word "road" and changing one letter at a time to form a new word, can you reach "tall" in four changes?

road
1. __
2. __
3. __
4. tall

(g) $6x - 4(5x + 9) = 4(2x - 9)$
(h) $5m - (3m - 7) + 2m + (2m - 3) = 14$
(i) $2(x - 3) + 3(x + 2) = 10$
(j) $4(2x - 1) - 3(2x + 2) = 0$

6. (a) $5(3x - 1) + 12 = 4(2x + 3) + 9$
(b) $3(2a - 5) - 7(a + 1) = 2(a + 1)$
(c) $3(2x - 4) - (x - 3) = 7(x + 2) - 1$
(d) $3(m - 7) - 5(m + 2) = -41$
(e) $-2(2m - 3) + 5(m + 7) = 40$
(f) $2(x^2 - 3x + 1) = 2(x^2 + x - 4) - 6$
(g) $2(a^2 + 3a + 2) - (2a^2 + 3a - 7) = 17$
(h) $12 = 2(x - 1) + 4(x + 2)$

7. Solve for x, correct to three significant figures.
(a) $1.2x + 0.3 = 1.74$
(b) $6.3a + 7.4 = 2.1a$
(c) $12x + 0.3 = 5x + 8.1$
(d) $3b + 0.7 + 0.2b = 14.1$
(e) $11.3 + 0.7a = 9.1$
(f) $9.8m + 6 = 7.2m - 6.3$

C **8.** Solve for x.
(a) $x + b = 7$
(b) $x - m = t$
(c) $x - d = -t$
(d) $ax = m$
(e) $-2x = a$
(f) $2x - a = m$
(g) $bx + t = m$
(h) $ax - m = t$
(i) $2bx - 3 = a$

2.7 LINEAR EQUATIONS PART II

In solving equations with fractional coefficients we first clear the fractions by multiplying both sides of the equation by the lowest common denominator (L.C.D.).

EXAMPLE 1. *Solve and check* $\frac{2}{5}x + \frac{x}{2} = 9$.

Solution

$$\frac{2}{5}x + \frac{x}{2} = 9$$

The L.C.D. of 5 and 2 is 10

$$10 \times \left[\frac{2}{5}x + \frac{x}{2}\right] = 10 \times [9]$$

$$4x + 5x = 90$$

$$9x = 90$$

$$x = 10$$

Check: L.S. $= \frac{2}{5}x + \frac{x}{2}$ R.S. $= 9$

$$= \frac{2}{5}(10) + \frac{10}{2}$$

$$= 4 + 5$$

$$= 9$$

$\therefore$ 10 is a root of the equation.

EXAMPLE 2. *Solve* $\frac{x-2}{3} - \frac{x+2}{2} - \frac{x-4}{4} = 0.$

Solution

$$\frac{(x-2)}{3} - \frac{(x+2)}{2} - \frac{(x-4)}{4} = 0$$

Bracket the numerators

$$12 \times \left[\frac{(x-2)}{3} - \frac{(x+2)}{2} - \frac{(x-4)}{4}\right] = 12 \times [0]$$

The L.C.D. for 2, 3 and 4 is 12

$$4(x-2) - 6(x+2) - 3(x-4) = 0$$

$$4x - 8 - 6x - 12 - 3x + 12 = 0$$

$$-5x - 8 = 0$$

$$-5x = 8$$

$$x = -\frac{8}{5}$$

The root is $-\frac{8}{5}$

EXERCISE 2-7

A Solve the following equations.

1. (a) $\frac{x}{3} = 10$ (b) $\frac{1}{2}m = 4$ (c) $\frac{2b}{3} = 6$

(d) $\frac{x}{5} = -6$ (e) $\frac{x}{2} = \frac{4}{3}$ (f) $\frac{m+1}{3} = 2$

B 2. (a) $\frac{x}{2} + \frac{x}{3} = 10$ (b) $\frac{x}{3} - \frac{1}{2} = \frac{1}{4}$

(c) $\frac{x}{4} - \frac{1}{5} = \frac{x}{2}$ (d) $\frac{3x}{5} + 1 = \frac{x}{4}$

(e) $\frac{b}{4} - \frac{b}{6} = 3$ (f) $\frac{2a}{3} - \frac{3a}{5} = \frac{1}{2}$

(g) $\frac{5y}{4} - 2y = 1\frac{1}{3}$ (h) $2m - \frac{1}{2} = \frac{m}{7} + \frac{m}{2}$

3. (a) $\frac{x+5}{2} + \frac{3x-3}{8} = -4$ (b) $\frac{x+2}{2} - \frac{x}{3} = 2$

(c) $\frac{2x+1}{3} = 6$ (d) $\frac{a-1}{3} = \frac{a}{6}$

(e) $\frac{m-1}{4} = \frac{3m-1}{6}$ (f) $\frac{m+7}{6} + \frac{1}{2} = \frac{m-2}{4} \quad \frac{1}{2}$

(g) $\frac{3x+7}{4} - \frac{x}{3} = -\frac{1}{3}$ (h) $\frac{2}{5}(3a-1) + \frac{1}{2}(a+2) = 4$

4. (a) $\frac{x}{2} - \frac{x+1}{3} = 6$ (b) $\frac{m-2}{3} - \frac{m+1}{2} = 4$

(c) $\frac{1}{2}(x + 3) - \frac{3x + 1}{4} = 2$ (d) $\frac{1}{3}(2x - 1) - \frac{1}{5}(4x - 6) = \frac{7}{15}$

(e) $\frac{x + 1}{2} - \frac{x - 2}{3} + \frac{x + 4}{4} = 3$ (f) $2\frac{1}{3} - \frac{3 - x}{6} + 4 = \frac{2x - 1}{2}$

2.8 EQUATIONS INVOLVING POLYNOMIALS

In this section the rules for simplifying polynomials and the rules for solving equations are used to solve equations containing polynomials.

EXAMPLE 1. *Solve and check*
$2(a - 3)(a + 2) - 15 = (2a - 1)(a + 2)$

Insert a mathematical symbol between 5 and 6 to make a number between 5 and 6.
5 ▨ 6

Solution

$$\begin{aligned}
2(a - 3)(a + 2) - 15 &= (2a - 1)(a + 2)\\
2(a^2 + 2a - 3a - 6) - 15 &= (2a^2 + 4a - a - 2)\\
2(a^2 - a - 6) - 15 &= (2a^2 + 3a - 2)\\
2a^2 - 2a - 12 - 15 &= 2a^2 + 3a - 2\\
2a^2 - 2a - 27 &= 2a^2 + 3a - 2\\
-2a - 3a &= -2 + 27\\
-5a &= 25\\
a &= -5
\end{aligned}$$

Check:
$$\begin{aligned}
\text{L.S.} &= 2(a - 3)(a + 2) - 15\\
&= 2(-5 - 3)(-5 + 2) - 15\\
&= 2(-8)(-3) - 15\\
&= 48 - 15\\
&= 33\\
\text{R.S.} &= (2a - 1)(a + 2)\\
&= (-10 - 1)(-5 + 2)\\
&= (-11)(-3)\\
&= 33
\end{aligned}$$

$\therefore$ -5 is a root of the equation.

EXERCISE 2-8

B Solve and check the following.

1. (a) $(k + 1)(k + 2) = k^2 - 9k + 26$
(b) $(x - 1)(x + 2) = x^2 - 5x + 6$
(c) $(m + 4)^2 = m(m + 6)$
(d) $(a - 3)(a + 4) - (a + 2)(a + 1) = -16$
(e) $(2y - 3)(y - 1) - 2(y + 2)(y + 3) = 6$
(f) $(2x - 3)(x - 2) - 2(x + 1)(x + 4) = 8 - 7x$

2. (a) $2(2x - 3) + (x - 4)(2x + 1) = 2x^2 - 1$
(b) $4x^2 - (2x - 1)(2x + 1) = x$
(c) $(a + 1)(a + 2) + (a - 1)(a - 3) = (2a - 1)(a + 1)$
(d) $(x - 5)(x - 2) = (x + 1)(x - 4)$
(e) $(x + 3)^2 = 2x(x + 1) - (x - 1)^2$
(f) $(m - 5)^2 + 3(m + 2)^2 = 4(m^2 - 1) - 7$

3. (a) $3(c-1)^2 - 3(c^2-1) = c - 15$
(b) $5(x+1)^2 + 7(x+3)^2 = 12(x+2)^2$
(c) $3(s-1)(s+2) - 2(s-4)^2 = s^2 + 19$
(d) $30 + 15(1-t) = (t-3)(t+15) - (t-3)^2$

2.9 SOLVING INEQUATIONS

Inequations are solved in the same manner as equations—with one exception. When both sides of the inequation are multiplied or divided by a negative number, the sense of the inequality must be reversed.

$4 < 8$
$4 + 3 < 8 + 3$
$4 - 7 < 8 - 7$
$2 \times 4 < 2 \times 8$
$4 \div 2 < 8 \div 2$
but
$4 < 8$
$(-1)4 \ ? \ (-1)8$
$-4 > -8$
and
$4 < 8$
$4 \div (-1) \ ? \ 8 \div (-1)$
$-4 > -8$

EXAMPLE 1. *Solve and graph the solution set* $\{x \mid 2x - 3 \leqq 15, x \in R\}$.

Solution
$$\begin{aligned} 2x - 3 &\leqq 15 \\ 2x &\leqq 18 && \text{Adding 3 to each side.} \\ x &\leqq 9 && \text{Dividing both sides by 2} \end{aligned}$$

The solution set is $\{x \mid x \leqq 9, x \in R\}$.

EXAMPLE 2. *Solve and graph the solution set* $\{x \mid 2(x-3) > 3x + 1, x \in R\}$.

Solution
$$\begin{aligned} 2(x-3) &> 3x + 1 \\ 2x - 6 &> 3x + 1 && \text{Remove brackets} \\ -x &> 7 \\ x &< -7 && \text{Reverse the sense of the inequality sign when dividing both sides by a negative number.} \end{aligned}$$

The solution set is $\{x \mid x < -7, x \in R\}$.

EXAMPLE 3. *Solve and graph the solution set*
$$\left\{x \mid \frac{x+1}{5} - 1 \leqq \frac{x-1}{2}, x \in R\right\}.$$

Solution

$$\frac{(x+1)}{5} - 1 \leqq \frac{(x-1)}{2}$$

Bracket the numerators.

$$10 \times \left[\frac{(x+1)}{5} - 1\right] \leqq 10 \times \left[\frac{(x-1)}{2}\right]$$

$$2(x+1) - 10 \leqq 5(x-1)$$

$$2x + 2 - 10 \leqq 5x - 5$$

$$2x - 8 \leqq 5x - 5$$

$$-3x \leqq 3$$

$$x \geqq -1$$

Note the change in the inequality sign.

The solution set is $\{x | x \geqq -1, x \epsilon R\}$.

EXERCISE 2-9

A

1. Solve the following inequations.

(a) $x + 3 < 6$ (b) $a - 2 > 6$ (c) $b - 3 \leqq 7$

(d) $m + 4 \geqq 8$ (e) $d - 5 < -4$ (f) $y + 2 \leqq -6$

(g) $3 + x < -4$ (h) $m - 3 \geqq -3$ (i) $2x > 6$

(j) $3m \leqq -6$ (k) $4b > -12$ (l) $-3x > 6$

(m) $-2a < -10$ (n) $-x < 10$ (o) $-3b \geqq 6$

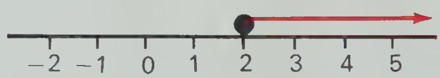
(p) $3 + x \leqq -7$ (q) $m + 4 < -8$ (r) $-3t > -27$

(s) $\frac{x}{3} > 6$ (t) $\frac{1}{2}a \leqq -8$ (u) $\frac{m}{4} \geqq -3$

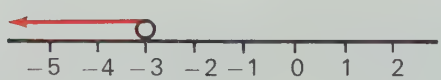
(v) $\frac{b}{-2} > 8$ (w) $\frac{c}{-3} \leqq -1$ (x) $\frac{a}{4} < 6 + 1$

2. State the solution set of each of the following.

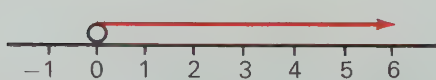
(a)



(b)

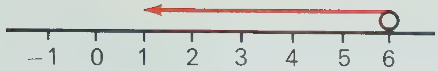


(c)

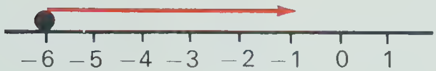


(d)

(e)

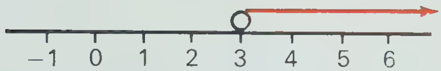


(f)



(g)

(h)



Test your skill:
$15.3 \div 0.003$

B **3.** Solve and graph the following. All variables have domain R.

(a) $2x + 3 < 9$ (b) $3a - 4 \geqq 8$ (c) $2m - 1 \leqq 9$
(d) $4b + 2 > 14$ (e) $3 + 2x \leqq 7$ (f) $3x + 2 \leqq 2x - 4$
(g) $4a - 1 < 2a + 7$ (h) $3m + 2 > 2m - 6$
(i) $4b - 3 - 2b > -7$ (j) $2x + 8 \leqq 0$

4. (a) $-2x + 3 < 7$ (b) $2x + 5 \geqq 3x - 3$
(c) $-3a + 7 > 2a - 3$ (d) $3 - x \leqq 2x + 6$
(e) $2(x - 2) \geqq 8$ (f) $3(x - 1) > 2(x + 3)$
(g) $4(a - 1) < 5(a + 2)$ (h) $2(m - 1) > 4(m + 2)$

5. (a) $2(x + 3) - 4 > 6$ (b) $2 + 3(m - 1) < 8$
(c) $2(m + 1) - 3(m + 2) \geqq -1$
(d) $3(b - 1) - (b + 2) > 3(b + 2)$
(e) $4(2x - 1) - 3 < 5(2x - 1)$
(f) $3 < 2(x - 1) + 1$
(g) $2x - (x - 1) + 3 \geqq 3(x + 2)$
(h) $4 - 3(a - 2) + 2a < 2a - 2$

6. (a) $\frac{x}{3} < \frac{1}{2}$ (b) $\frac{x}{4} - 1 \leqq \frac{x}{3} - 2$

(c) $\frac{x - 1}{2} > 3$ (d) $\frac{a + 2}{3} \geqq -1$

(e) $\frac{a - 1}{2} - 3 > \frac{a + 2}{4}$ (f) $\frac{2x - 1}{3} - 1 \geqq \frac{1 + x}{2}$

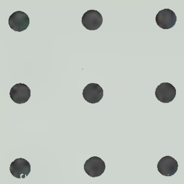
(g) $\frac{2m + 1}{4} > \frac{3m - 1}{5} + \frac{1}{4}$ (h) $\frac{3x - 2}{2} < \frac{4x + 1}{5} + \frac{1}{5}$

2.10 FORMULA SOLVING AND SUBSTITUTION

In formulas, the variable being evaluated is often not the "subject" of the formula. It is convenient to be able to restate the formula so that the variable being evaluated is the "subject".

EXAMPLE 1. *Find the value of h when $I = 16$ and $c = 32$, where*

$$I = \frac{8h - c}{3}$$



Cross all dots with four straight lines without taking your pencil off the paper.

Solution

Method I (substitution first)

$$l = \frac{8h - c}{3}$$

$$16 = \frac{8h - 32}{3}$$

$$48 = 8h - 32$$

$$80 = 8h$$

$$10 = h$$

Method II (restate the formula in terms of h first)

$$l = \frac{8h - c}{3}$$

$$3l = 8h - c$$

$$3l + c = 8h$$

$$\frac{3l + c}{8} = h$$

or $h = \dfrac{3l + c}{8}$

$$= \frac{3(16) + 32}{8}$$

$$= \frac{80}{8}$$

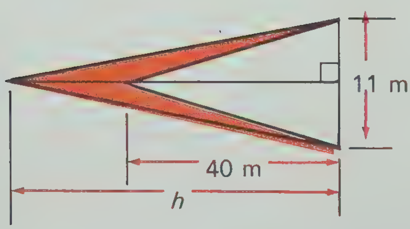
$$= 10$$

EXERCISE 2-10

B

1. (a) If $V = lwh$, find V when $l = 10$, $w = 5$, $h = 2$.
(b) If $A = \frac{1}{2}bh$, find A when $b = 20$, $h = 18$.
(c) If $V = lwh$, find w when $l = 12$, $h = 4$, $V = 144$.
(d) If $A = \frac{1}{2}bh$, find b when $h = 8$, $A = 40$.
(e) If $P = 2(l + w)$, find l when $P = 146$, $w = 13$.
(f) If $S = \frac{n}{2}(a + l)$, find l when $S = 590$, $n = 20$, $a = 1$.
(g) If $s = ut + \frac{1}{2}at^2$, find u when $s = 32$, $t = 4$, $a = 2$.
(h) If $A = p + prt$, find t when $A = 3500$, $p = 500$, $r = 3$.

2. (a) If $V = lwh$, find V when $l = 9.2$, $w = 3.3$, $h = 4$.
(b) If $P = 2(l + w)$, find P when $l = 3.7$, $w = 1.8$.
(c) If $A = \frac{1}{2}bh$, find b when $A = 62.4$, $h = 3.9$.
(d) If $V = lwh$, find h when $V = 0.0024$, $l = 0.06$, $w = 0.02$.
(e) If $P = 2(l + w)$, find l when $P = 18.4$, $w = 1.7$.
(f) If $C = 2\pi r$, find r when $C = 44.588$, $\pi = 3.14$.



3. The area of the shaded figure is given by the formula $A = \dfrac{11(h - 40)}{2}$ where A is the area in square metres and h is the height in metres. Find the height when the area is 160 m².

4. The area of a trapezoid is given by $A = \dfrac{h}{2}(x + y)$. If $A = 84$ cm², $h = 8$ cm and $x = 5$ cm, find y.

5. The formula for true interest rate is given by $r = \dfrac{2NC}{P(n + 1)}$ where C is

the carrying charge in dollars, N is the number of payment intervals in a year, P is the principal or cash price in dollars, and n is the total number of interest payments. Find C, the carrying charge, if $N = 12$, $P = \$150$, $n = 12$ and $r = 0.04$.

6. Solve each of the following formulas for the variable indicated.

(a) $p = 4t$, for t
(b) $V = lwh$, for w
(c) $d = vt$, for v
(d) $V = IR$, for I
(e) $pv = Rt$, for R
(f) $P = 2(l + w)$, for l
(g) $S = \frac{n}{2}(a + l)$, for a
(h) $s = ut + \frac{1}{2}at^2$, for a
(i) $A = p + prt$, for r

REVIEW EXERCISE

A

Simplify

1. (a) $\frac{26abc}{13ab}$ (b) $\frac{16a^2}{-4a}$ (c) $\frac{-20mnt}{-5t}$
(d) $\frac{100a^2b^2c}{-10a^2c}$

2. Solve

(a) $x + 3 = 10$ (b) $a - 12 = 0$ (c) $2t = -10$
(d) $-4b = -12$ (e) $3 + m = -4$ (f) $3x + 1 = 10$
(g) $\frac{a}{2} = 4$ (h) $\frac{y}{5} = -10$ (i) $2a - 2 = 8$

3. Expand the following.

(a) $4(x^2 + 3x + 7)$ (b) $-2(a - 2b)$ (c) $-(3a - 1)$
(d) $2x(x^2 - x)$ (e) $-3(2m^2 - 2m + 1)$
(f) $-2x(x - y)$ (g) $4(-3a - 2b)$ (h) $-3ab(2a - 3b)$
(i) $-(-3x^2 - 2x + 7)$

B

4. Expand and simplify.

(a) $3(a - 7) - 4(a + 3)$
(b) $2(x - 3) - (x + 7)$
(c) $-4(a - 3b + c) - 2(2a - b)$
(d) $2a(3a - 1) - a(a + 2)$
(e) $4m^2 - m(2m - 1) + 3m$
(f) $-3x(x + y) + 2x(2x - y)$

5. Expand

(a) $(a - 3)(a + 2)$
(b) $(3a - 1)(a + 2)$
(c) $(1 - 3x)(x + 1)$
(d) $(2x - 1)^2$
(e) $(x + 3)(x - 7)$
(f) $(2x + 5)(3x - 4)$
(g) $(2b - 3a)^2$
(h) $(1 - 4x)(1 + 4x)$

6. Simplify the following.

(a) $2(x + 7) + (x - 3)(x + 2)$
(b) $2(a - 1)(a + 1) - (a + 3)(a + 4)$
(c) $3(b + 2)^2 - 2(b - 1)(b + 2)$
(d) $(1 - 2x)(1 + x) - (1 - x)^2$
(e) $4(2t - 1)(t + 2) - 3(1 - 2t)(1 + t)$
(f) $(a + 2b)^2 - 3(a - b)(a + b)$

7. Simplify

(a) $\dfrac{2x + 6}{2}$ (b) $\dfrac{-7x^2 + 14x}{-7}$ (c) $\dfrac{10ab - 6bc}{2b}$

(d) $\dfrac{100a^2 - 60a^3}{-10a}$ (e) $\dfrac{3m - 6n + 9t}{-3}$

(f) $\dfrac{12x^4 + 6x^3 - 6x^2}{6x^2}$ (g) $\dfrac{-12a^2b + 6ab^2 - 6ab}{6ab}$

8. Divide

(a) $(a^2 + 9a + 18) \div (a + 6)$
(b) $(x^2 - x - 12) \div (x - 4)$
(c) $(m^2 + 7m + 13) \div (m + 4)$
(d) $(x^3 + 5x^2 + 7x + 3) \div (x + 3)$
(e) $(11a - 10a^2 + 2a^3 - 12) \div (a - 4)$
(f) $(-m^2 + 6m^3 - 11m - 7) \div (3m + 2)$
(g) $(r^3 - 9) \div (r + 3)$
(h) $(3x^2 - 8x + 6x^3 - 4) \div (1 + 2x)$

9. Solve and check.

(a) $3x + 7 = 2x - 4$ (b) $5b = 7b - 4$
(c) $2(x - 1) = x + 6$ (d) $3(m + 1) + 4 = 2(m - 1)$
(e) $2(x + 3) + 3(x - 1) = 3(x - 2) + 1$
(f) $5 + 2(b - 3) - (b - 7) + 6 = 11$
(g) $3 + 2(2x - 1) + 6 = 5(x - 3)$
(h) $3a - 2(a - 3) + 7 = 4a - (a - 1)$
(i) $4x - (3x - 1) - 3 + 6(x - 2) = 0$

10. Solve

(a) $\dfrac{2x}{3} = \dfrac{3}{2}$ (b) $\dfrac{a}{2} + 1 = \dfrac{a}{3}$ (c) $\dfrac{b + 1}{2} = 4$

(d) $\dfrac{2m - 1}{3} = 5$ (e) $\dfrac{a + 3}{2} - 1 = 0$

(f) $\dfrac{x + 2}{2} = \dfrac{x - 1}{3}$ (g) $\dfrac{b + 7}{4} = \dfrac{2b - 1}{3}$

(h) $\dfrac{1}{2}(x + 1) + \dfrac{1}{3}(x + 1) = 5$

(i) $\dfrac{1}{3}(2m + 1) - \dfrac{1}{4}(m + 1) = 3$

(j) $\dfrac{1 - 3a}{4} - \dfrac{a - 1}{3} = -a$

11. Solve and check.

(a) $(a + 1)(a + 4) = (a + 2)(a + 1)$
(b) $(x - 3)(x + 2) - (x + 3)(x + 1) = 1$
(c) $(b - 3)(2b + 1) = 2(b - 1)(b + 1) + 9$
(d) $2(x + 3)(x + 2) - 2(x - 6) = 2x^2$
(e) $(2a - 3)(a - 2) - 2(a^2 - 1) = -6$
(f) $2(b + 1)(b + 2) - (2b - 1)(b + 3) = 6$
(g) $3(2x - 3)(x + 2) = (3x - 1)(2x - 1) + 5$

12. Solve and graph the solution set.

(a) $2x + 1 \leqq 3$

(b) $4x - 3 > 2x + 1$

(c) $3x + 5 < 5x - 3$

(d) $3(x - 1) + 9 > 0$

(e) $2(x - 1) + 3 \geqq 4(1 + x) + 1$

(f) $2 < 2(x - 1) + 3(x + 3)$

(g) $\frac{a}{2} + 1 > \frac{3}{4}$

(h) $\frac{b - 1}{3} \leqq 1$

(i) $\frac{4x - 2}{5} - \frac{3x - 1}{2} > 5$

(j) $\frac{m - 3}{2} - \frac{m + 1}{3} < -1$

13. (a) If $A = lw$, find A when $l = 5$ and $w = 3$.

(b) If $A = lw$, find l when $A = 40$ and $w = 4$.

(c) If $P = 2(l + w)$, find l when $P = 30$ and $w = 5$.

(d) If $A = \frac{h}{2}(a + b)$, find b when $A = 40$, $h = 10$, and $a = 3$.

(e) If $A = \frac{1}{2}bh$, find h when $A = 50$ and $b = 10$.

(f) If $I = prt$, find p when $I = 400$, $r = \frac{1}{10}$ and $t = 20$.

REVIEW AND PREVIEW TO CHAPTER 3

EXERCISE 1

1. Complete the following to form true statements.

(a) $\frac{2}{5} = \frac{\square}{20} = \frac{10}{\square}$

(b) $\frac{5}{8} = \frac{\square}{16} = \frac{40}{\square}$

(c) $\frac{\square}{8} = \frac{21}{24} = \frac{\square}{40}$

(d) $\frac{3}{7} = \frac{9}{\square} = \frac{24}{\square}$

(e) $\frac{\square}{6} = \frac{4}{24} = \frac{\square}{30}$

(f) $\frac{\square}{2} = 2 = \frac{14}{\square}$

(g) $3\frac{1}{2} = \frac{\square}{2} = \frac{\square}{8}$

(h) $2\frac{2}{3} = \frac{8}{\square} = \frac{\square}{9}$

(i) $\frac{\square}{20} = \frac{9}{5} = \frac{45}{\square}$

2. Replace $\square$ with $>$ or $<$ to form a true statement.

(a) $\frac{3}{5} \square \frac{4}{5}$	(b) $2\frac{1}{3} \square 2\frac{1}{2}$	(c) $\frac{2}{3} \square \frac{4}{7}$
(d) $\frac{3}{5} \square \frac{5}{9}$	(e) $\frac{13}{8} \square 1\frac{2}{3}$	(f) $2\frac{2}{5} \square \frac{16}{7}$
(g) $\frac{13}{16} \square \frac{4}{5}$	(h) $5\frac{3}{5} \square \frac{38}{7}$	

EXERCISE 2

3. Simplify

(a) $\frac{3}{4} \times \frac{1}{2}$	(b) $\frac{7}{8} \times \frac{5}{6}$	(c) $\frac{4}{7} \times \frac{3}{8}$	(d) $1\frac{1}{2} \times \frac{3}{4}$
(e) $6\frac{1}{2} \times 3\frac{3}{5}$	(f) $2\frac{3}{4} \times 1\frac{1}{8}$	(g) $5\frac{1}{5} \times 2\frac{1}{3}$	(h) $\frac{11}{4} \times 2\frac{3}{5}$
(i) $2 \times 3\frac{1}{8}$	(j) $4\frac{2}{3} \times 4$	(k) $\frac{11}{6} \times 3\frac{1}{2}$	

4. Simplify

(a) $2\frac{1}{4} \div \frac{1}{2}$	(b) $3\frac{1}{3} \div 2\frac{5}{6}$	(c) $4 \div 2\frac{1}{4}$	(d) $3\frac{1}{8} \div 2$
(e) $\frac{11}{3} \div \frac{11}{5}$	(f) $2\frac{3}{4} \div 1\frac{5}{8}$	(g) $4 \div 3\frac{1}{6}$	(h) $2\frac{7}{8} \div 3\frac{3}{4}$
(i) $6 \div 2\frac{1}{9}$	(j) $4\frac{1}{5} \div 3\frac{5}{6}$	(k) $2\frac{7}{8} \div 1\frac{1}{2}$	

EXERCISE 3

5. Simplify

(a) $\frac{2}{3} + \frac{3}{4}$	(b) $1\frac{1}{2} + 2\frac{3}{5}$	(c) $4\frac{1}{6} + 3\frac{1}{2} + \frac{1}{4}$
(d) $2\frac{1}{8} + 3\frac{1}{6} + \frac{3}{4}$		(e) $5\frac{3}{5} + 2\frac{1}{10} + 3\frac{1}{20}$
(f) $4\frac{3}{8} + 7\frac{5}{6} + \frac{1}{3}$		(g) $\frac{22}{3} + 5 + 6\frac{1}{4}$
(h) $\frac{12}{5} + 3\frac{7}{8} + 7\frac{17}{20}$		(i) $22 + 3\frac{5}{6} + \frac{23}{3}$

6. Simplify

(a) $2\frac{3}{4} - 1\frac{1}{8}$ (b) $3\frac{4}{5} - 2\frac{1}{4}$ (c) $5\frac{1}{2} - 4\frac{5}{8}$

(d) $7 - 3\frac{3}{8}$ (e) $\frac{25}{4} - 1\frac{1}{2}$ (f) $16\frac{1}{8} - 3\frac{4}{5}$

(g) $\frac{33}{4} - 2\frac{1}{2}$ (h) $11 - 3\frac{7}{8}$ (i) $6\frac{1}{5} - 2\frac{3}{4}$

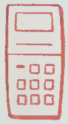
7. Perform the operations indicated.

(a) $2\frac{1}{2} \times \frac{3}{5} \times \frac{7}{3}$ (b) $1\frac{1}{4} \div (\frac{3}{5} \times \frac{1}{2})$ (c) $\frac{3}{4} \times (\frac{1}{5} + 1\frac{1}{3})$

(d) $(1\frac{1}{7} - \frac{5}{6}) \times \frac{7}{8}$ (e) $5 \times \frac{2}{9} \div 1\frac{1}{3}$ (f) $(\frac{1}{2} + \frac{1}{3}) \times (\frac{3}{4} - \frac{5}{8})$

(g) $(\frac{11}{8} + 1\frac{1}{2}) \div 3$ (h) $4 \div (1\frac{1}{4} - \frac{3}{7})$ (i) $\frac{2}{3} \div 1\frac{1}{6} \times 2\frac{1}{4}$

Solve the following equations.



1. $7.813x = 8.143$

2. $21.44x + 53.75 = 81.46$

3. $7.816x - 7.213 = 1.473$

4. $0.9873x + 0.4182 = 0.6376$

5. $-83.42x = -162.43$

6. $3.814 - 1.753x = 8.593$

7. $82.43x - 66.31x = 97.43$

8. $9.417x + 1.763 = 6.431x - 9.743$

9. $(16.43 - 12.77)x = 81.44$

10. $53.75x + 48.17 = 77.59 - 16.34x$

Equation of the Straight Line

3.1 SLOPE

The slope of a straight line is the measure of the direction of the line.

Line AB has a greater slope (steepness) than line CD.

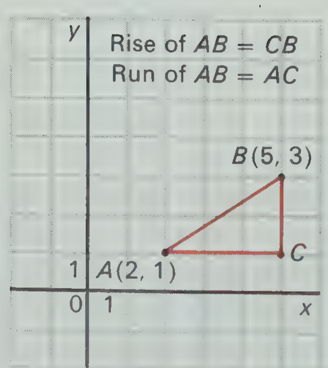
The slope is defined as the quotient of the vertical change (called the rise) divided by the horizontal change (called the run).

$$\text{SLOPE} = \frac{\text{RISE}}{\text{RUN}}$$

EXAMPLE 1. *Find the slope of the line segment from the point A(2, 1) to the point B(5, 3).*

Solution The vertical change or rise of AB is 2 (that is, 3 subtract 1), while the horizontal change or run of AB is 3 (5 subtract 2). Therefore:

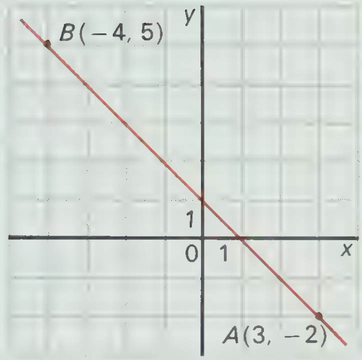
$$\text{Slope } AB = \frac{\text{rise } AB}{\text{run } AB} = \frac{3-1}{5-2} = \frac{2}{3}$$



We use $\triangle y$ to denote the change in y and $\triangle x$ to denote the change in x. We use the letter m to denote slope.

$$\boxed{\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\triangle y}{\triangle x} = m}$$

EXAMPLE 2. *Find (a) slope AB; and (b) slope BA.*



Solution To find the slope of AB, $\triangle x = (-4) - 3 = -7$ and

$$\triangle y = 5 - (-2) = 5 + 2 = 7$$

$$\text{so } m = \frac{\triangle x}{\triangle y} = -\frac{7}{7} = -1$$

To find the slope of BA

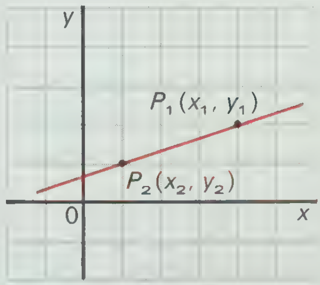
$$m = \frac{\triangle y}{\triangle x} = \frac{(-2) - 5}{3 - (-4)}$$

$$= \frac{-2 - 5}{3 + 4} = \frac{-7}{7} = -1$$

The slope of AB is the same as the slope of BA. Remember, the slope of the hill to be climbed is the same as the slope of the hill to be walked down.

The run from A to B is -7 while the run from B to A is 7. This indicates different directions. The rise from A to B is 7 (going up) while the rise from B to A is -7 (going down). Slope AB = Slope BA.

EXAMPLE 3. *Find the slope of the line segment joining $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.*



Solution $\triangle y =$ (the y-coordinate of P_2) − (the y-coordinate of P_1) $= y_2 - y_1$.
$\triangle x =$ (the x-coordinate of P_2) − (the x-coordinate of P_1) $= x_2 - x_1$.
Representing the slope of P_1P_2 by m,

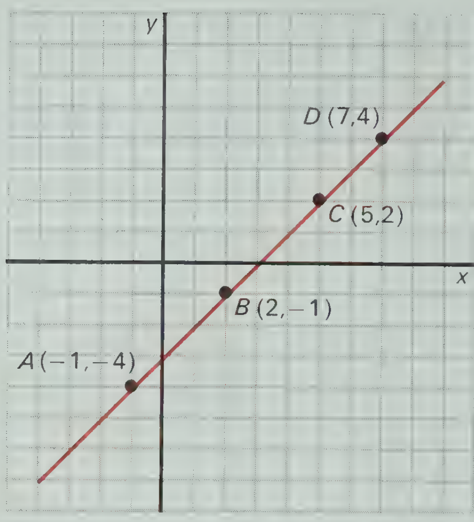
$$\boxed{m = \frac{\triangle y}{\triangle x} = \frac{y_2 - y_1}{x_2 - x_1}}$$

The slope of the line segment from any point $P_1(x_1, y_1)$ to any point $P_2(x_2, y_2)$ is $\frac{y_2 - y_1}{x_2 - x_1}$.

INVESTIGATION 3.1

1. A, B, C, D are four points lying on a straight line.
(a) Complete the table below.

Segment	$\triangle y$	$\triangle x$	$\frac{\triangle y}{\triangle x}$
AB			
AC			
BC			
BD			
CD			



(b) What do you notice about $\frac{\triangle y}{\triangle x}$ for different pairs of points on the straight line?
(c) What can be said about the slope of a straight line with respect to the slope of any segment of the line?

2. (a) Find the slope of the line segment from:
(i) $R(-4, -5)$ to $S(0, 1)$ (ii) $R(-4, -5)$ to $T(4, 7)$
(iii) $S(0, 1)$ to $T(4, 7)$
(b) What can you say about the three points R, S, and T?

3. (a) Find the slope of the line segment joining $A(-2, -1)$ to $B(3, 2)$.
(b) What sign has the slope of a line segment which rises towards the right?

4. (a) Find the slope of the line segment joining $C(5, -2)$ to $D(-1, 3)$.
(b) What sign has the slope of a line segment which rises towards the left?

Insert signs to make
3 ▨ 2 ▨ 7 ▨ 4 ▨ 1 = 9
a true statement.

5. (a) Find the slope of a line through $E(3, -2)$ and $F(-3, -2)$.
(b) What is the slope of a line parallel to the x-axis?
(c) Does this answer seem reasonable in connection with our everyday use of the word slope?

6. (a) Find the slope of a line through $E(3, -2)$ and $G(3, 5)$.
(b) What is the slope of a line parallel to the y-axis?
(c) Does this answer seem reasonable in connection with our everyday use of the word slope?

EXERCISE 3-1

B **1.** Determine the slope of the line which passes through each of the following sets of points.
(a) (6, 8) and (10, 5) (b) (10, 2) and (16, −1)
(c) (0, 0) and (3, 5) (d) (0, 2) and (0, −3)
(e) (−2, 0) and (1, 0) (f) (3, 5) and (−2, −1)
(g) (−2, −4) and (3, −5) (h) (−5, 1) and (3, −6)

(i) $(7, -6)$ and $(0, -3)$ (j) $(-5, 1)$ and $(4, -3)$
(k) $(-8, -1)$ and $(-6, -3)$ (l) $(-1, -1)$ and $(-5, 6)$

2. (a) When calculating the slope of a line, does the order in which the points are named make any difference?
(b) When calculating $\triangle x$ and $\triangle y$, does the order in which the points are named make any difference?
(c) What is the slope of a line:
(i) parallel to the x-axis? (ii) parallel to the y-axis?
(d) State the sign of the slope of a line which:
(i) rises to the left. (ii) rises to the right.

Addition:
SEND
MORE
MONEY

3. (a) Calculate the slope of $J(-4, 2)$, $K(5, -2)$.
(b) Calculate the slope of $K(5, -2)$, $J(-4, 2)$.
(c) Compare the slopes.

4. By finding the values of the run and the rise of each of the following, state where the second point is located relative to the first.
(a) $(4, 7)$ to $(2, -3)$ (b) $(-3, 4)$ to $(4, 1)$
(c) $(3, 4)$ to $(-2, 4)$ (d) $(-2, -3)$ to $(2, -1)$
(e) $(-1, 7)$ to $(3, 2)$ (f) $(-6, 2)$ to $(-6, -7)$

C **5.** A line through $(-4, 3)$ and $(x_2, 6)$ has a slope of $\frac{3}{4}$. Find x_2.

6. Find p if the slope of the line through $(p, 2)$ and $(1, 0)$ is the same as the slope of the line through $(-2, 1)$ and $(2, 3)$.

7. Find the point on the y-axis such that the line segment from this point to $(4, 7)$ has slope $\frac{5}{2}$.

8. (a) Use the slope formula to show that the points $X(-2, -2)$, $Y(2, 1)$, $Z(6, 4)$ lie on a straight line.
(b) Determine the value of a such that $P(a, 5)$, $Q(1, 3)$, $R(-2, 1)$ lie on a straight line.

9. A switchback on a highway through the mountains is to have a slope of $\frac{1}{20}$. What should the rise be over a run of 1000 m?

3.2 THE EQUATION OF A LINE

INVESTIGATION 3.2

1. Plot six points for which the value of the x-coordinate is the same as the value of the y-coordinate; that is, for which $x = y$. What seems to be true of these points? Join these points with the best possible graph.

2. Fill in the table below so that the pairs satisfy the condition that the y-coordinate be the opposite of the x-coordinate.

x	6	3	-2			1.5
y	-6			0	-4	

Plot these points. What kind of graph can be drawn through all these points? Extend the graph as far as possible in both directions. Is the point $(7, -7)$ on this line? What is the relation which describes this line?

3. Plot five points for which the y-coordinate is twice the x-coordinate. If the points are joined what type of graph do you get? Name the relation describing this graph.

4. For the relation $\{(x, y) \mid y = 2x + 3, x, y \in R\}$, complete the following table of values.

x	0	1	3	-2
y	■	■	■	■

Plot the corresponding points and join. What type of graph has been formed?

5. Make a table of values showing three ordered pairs for each of the following and draw a graph.
(a) $y = 3x - 1$ (b) $y = \frac{2}{3}x + 4$ (c) $y = -2x + 5$
(d) Why are three ordered pairs sufficient to draw these graphs?

6. (a) What type of graph was obtained in each of the first five questions?
(b) What is the form of the defining relation of each of these graphs?
(c) Equations such as these are called *linear* equations. Can you suggest why?

7. For the relation $\{(x, y) \mid y = x^2, x, y \in R\}$, complete the following table. Plot the corresponding points and draw a graph. Is this graph a straight line?

x	4	3	2	1	0	-2	-3
y	16	■	■	■	■	■	■

8. Make a table of values showing five ordered pairs for each of the following and draw a graph.
(a) $y = x^2 + 2$ (b) $y = x^3$

9. (a) Are the graphs obtained in questions 7 and 8 straight lines?
(b) Are the defining equations linear?

So far in our study of the straight line we have investigated three ideas.
(1) The basic property of a straight line, namely:

Slopes of all segments of ■ straight line are equal.

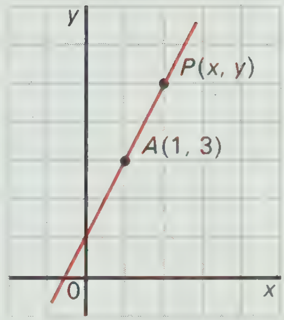
(2) A line is a set of points such that the slopes of all possible pairs of points from the set are equal.
(3) Just as a point is named by an ordered pair of numbers, a line is named by a linear equation.

We use these three ideas to find out more about the equation of a line.

EXAMPLE 1. *Find the equation of the line passing through the point A(1, 3) with slope 2.*

Solution Let $P(x, y)$ be any other point on the line. By the basic slope property of a straight line, slope $AP = 2$. Therefore

$$\frac{y-3}{x-1} = 2 \text{ and } y - 3 = 2(x-1)$$



Sometimes it is more convenient to simplify the equation and put it in the standard form—the x term, then the y term, then the constant term, equals zero; that is, in the form $Ax + By + C = 0$, where $A, B, C \in R$. Remember that x and y always stand for the coordinates of any point on the line. We say that a line "passes through a point" or a point "lies on a line" if the coordinates of the point satisfy the equation of the line. By the same token, if a point does lie on a line then its coordinates must satisfy the equation of the line.

EXAMPLE 2. *Write the equation $y - 3 = 2(x - 1)$ in the form, $Ax + By + C = 0$.*

Solution

$$\begin{aligned} y - 3 &= 2(x-1) \\ \therefore y - 3 &= 2x - 2 \\ \therefore 2x - 2 - y + 3 &= 0 \\ \therefore 2x - y + 1 &= 0 \end{aligned}$$

EXAMPLE 3. *Find the equation of the line through $P_1(x_1, y_1)$ with slope m.*

Solution Let $P(x, y)$ be any other point on this line. Therefore the slope of

$$P_1P = m,$$

$$\frac{y - y_1}{x - x_1} = m, \text{ and}$$

$$\boxed{y - y_1 = m(x - x_1)}$$

This form of a line equation is called the **point-slope form** of the line equation.

EXAMPLE 4. *Find the equation of the line passing through $B(-1, 2)$ and $C(2, -4)$.*

Solution The slope of the line through BC is

$$m_{BC} = \frac{\triangle y}{\triangle x} = \frac{2-(-4)}{-1-2} = -2$$

Using the point $(2, -4)$ and the formula

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-4) &= -2(x-2) \\ y + 4 &= -2x + 4 \\ 2x + y &= 0 \end{aligned}$$

Starting with the word "car" and changing one letter at a time to form a new word, can you reach "tip" in three changes?

car
1. —
2. —
3. tip

EXERCISE 3-2

A

1. State the slope of the line determined by the following pairs of points.
(a) $(5, 4), (3, -1)$
(b) $(1, 3), (7, 2)$
(c) $(0, -3), (4, 0)$
(d) $(2, -2), (-2, 2)$
(e) $(-4, 3), (3, -1)$
(f) $(2, -6), (-5, -1)$

2. State in unsimplified form the equation of each line represented in question 1.

3. State the equation of the line in unsimplified form for each case below.

	(a)	(b)	(c)	(d)	(e)	(f)
Point	$(-5, 2)$	$(2, -2)$	$(0, 3)$	$(-8, 2)$	$(4, 1)$	$(0,0)$
Slope	3	-1	$\frac{1}{2}$	$\frac{2}{3}$	0	$-\frac{4}{3}$

B

4. Write the equation of each line determined by the following information in the form $Ax + By + C = 0$.
(a) Passing through $(7, 3)$ with slope $\frac{5}{3}$
(b) Passing through $(0, 9)$ and $(-8, 1)$
(c) Passing through $(1, 1)$ and $(3, 4)$
(d) Passing through $(-2, 5)$ and having slope $-\frac{3}{2}$

5. What is the equation of a line passing through the origin with slope:
(a) 1
(b) -5
(c) m

6. (a) Find the slope of $R(3, 6)$, $S(7, 6)$.
(b) Plot these points on a graph and draw a straight line through them.
(c) How would you describe this line?
(d) Find the equation of RS.
(e) Repeat parts (a) to (d) for $U(-3, 5)$, $V(11, 5)$.

7. Find the equation of a line passing through $P_1(x_1, y_1)$ and parallel to the x-axis.

8. (a) Name two points on the x-axis.
(b) What is the slope of the x-axis?
(c) What is the equation of the x-axis?

9. (a) Find the slope of $S(7, 6)$, $T(7, -2)$.
(b) Plot these points on a graph and draw a straight line through them.
(c) How would you describe this line?
(d) Find the equation of ST.
(e) Repeat parts (a) to (d) for $V(11, 5)$, $W(11, -5)$.

10. Find the equation of the line passing through $P_1(x_1, y_1)$ and parallel to the y-axis.

11. (a) Name two points on the y-axis.
(b) What is the slope of the y-axis?
(c) What is the equation of the y-axis?

12. Find the equation of the line through the point $(3, 4)$ and:
(a) with slope $\frac{1}{2}$
(b) through $(-5, -2)$

(c) through (0, 0) (d) parallel to the x-axis
(e) parallel to the y-axis

13. Write the equation of the line with slope:
(a) 2 and passing through (0, 1)
(b) 7 and passing through (0, −4)
(c) −3 and passing through (0, b)
(d) m and passing through (0, b)

14. State the equations of the lines determined by the following information.

Point	(2, 0)	(−3, 0)	(11, 0)	(a, 0)
Slope	5	$\frac{1}{2}$	m	m

15. (a) Find the equation of the straight line which cuts the x-axis at a distance of 4 units from the origin and cuts the y-axis at a distance 3 units from the origin.
(b) Find the equation of the line through (a, 0) and (0, b).

Test your skill:
$0.0004 \div 5$

16. (a) Write the equation of each line formed in question 4 in the form $y = mx + b$.
(b) Draw the graph of each line.
(c) By comparing the answers in (a) to the graphs in (b), what does the m in $y = mx + b$ seem to represent?
(d) What does the b represent?

3.3 THE ROLES OF m AND b IN $y = mx + b$

INVESTIGATION 3.3

1. (a) Using the same set of axes, draw graphs representing relations having the following defining equations:
(i) $y = 2x$ (ii) $y = 2x + 3$
(iii) $y = 2x + 5$ (iv) $y = 2x - 3$
(b) In each of these equations, written in the form $y = mx + b$, what are the values of m and b?
(c) How are the graphs of these four lines related to each other?
(d) By choosing two points on each line, find the slope of each line.
(e) What can you conclude about the slopes of lines which are related such as those you have found in (c)?
(f) How is the slope of a line related to the equation of the line?
(g) What is the slope of the line represented by $y = mx + b$?

2. (a) State the slope of each line whose defining equation is given as:
(i) $y = 3x + 1$ (ii) $y = 4x - 5$ (iii) $y = -2x + 7$
(iv) $y = \frac{1}{2}x + 1$ (v) $y = 4$ (vi) $y = -\frac{1}{5}x - 2$
(b) Describe each line in (a) as "rising upward to the right", "rising upward to the left", or "parallel to the x-axis".
(c) How is the slope of a line related to its steepness?

3. (a) Each of the four equations in question 1 could have been written in the form $y = 2x + b$. What substitution would have had to be made for b in order to get the original equations? Such a set of lines is called a **family of lines** because each member of the family has a common characteristic, in this case a slope of 2. $y = 2x + b$ represents a family of parallel lines with slope 2.
(b) Describe each of these families of lines.
(i) $y = 3x + b$ (ii) $y = -\frac{1}{2}x + b$

4. (a) Using the same set of axes, draw graphs representing relations having the following defining equations:
(i) $y = x + 5$ (ii) $y = \frac{2}{3}x + 5$
(iii) $y = -3x + 5$ (iv) $y = 5$
(b) In each of these equations, written in the form $y = mx + b$, what are the values of m and b?
(c) How are the graphs of these four lines related to each other?
(d) The distance from the origin to the point where a line cuts the y-axis is called the **y-intercept** of the line; that is, the y-intercept of a line is the y-coordinate of the point of intersection of the line and the y-axis. What is the y-intercept of each line in (a)?
(e) What is the y-intercept of $y = mx + b$?

KNOCK-OUT
16 teams enter a basketball tournament. Each team will play until it loses. Assuming no ties, how many games are required to produce a winner?

5. State the y-intercept of each line in 2(a).

6. (a) Each of the four equations in question 4 could have been written in the form $y = mx + 5$. What substitution would have had to be made for m in order to get the original equations?
(b) What is the common characteristic of the family of lines $y = mx + 5$?
(c) Describe each of these families of lines.
(i) $y = mx + 4$ (ii) $y = mx - \frac{1}{3}$

7. Complete the following table in your workbook.

Equation	$y = mx + b$	Slope	y-intercept	Coordinates of one point on line
$2x + y = 6$				
$x + 2y = 3$				
$2x + 3y = 1$				
$y = 3$				
$2x + 5y = 0$				

8. (a) State the slope and y-intercept of $y = 3x + 4$.
(b) What are the coordinates of one point on $y = 3x + 4$?
(c) Plot this point.
(d) Since the slope is known, and recalling that $m = \frac{\triangle y}{\triangle x}$ if $\triangle x = 1$, what must $\triangle y$ be?

(e) Name and then plot a second point on $y = 3x + 4$.
(f) Graph $y = 3x + 4$.

9. Graph $y = 2x + 3$ by the method of question 8.

EXERCISE 3-3

B **1.** Write the equation of each of the following lines.
(a) Through (1, 2) with slope 3
(b) Through (−3, 2) with slope 4
(c) Through (0, 0) with slope 1
(d) Through (0, 4) with slope −7

2. Write the equation of each of the following lines.
(a) Through (1, 1) and (3, 4)
(b) Through (−1, −5) and (−5, 7)
(c) Through (0, 0) and (7, −2)
(d) Through (0, 4) and (5, 0)

3. Write the equation of each of the following lines.
(a) With slope 3 and y-intercept 8
(b) With slope $\frac{3}{4}$ and y-intercept $-\frac{2}{3}$
(c) With slope $-\frac{4}{5}$ and y-intercept $-\frac{7}{3}$
(d) With slope 0 and y-intercept 5

> $y = mx + b$ is called the **slope y-intercept form** of a straight line equation.

4. (a) How would you define an **x-intercept**?
(b) If the x-intercept of a line is 2, name one point on the line.
(c) If the same line has slope 3, find the equation of the line.
(d) Show that the equation of a line with slope m and x-intercept a is $y = m(x - a)$.

5. (a) By changing the following equations to the form $y = mx + b$, determine the slope and the y-intercept.
(i) $2x + y = 5$
(ii) $2x + 3y = 4$
(iii) $3x + 2y + 6 = 0$
(iv) $2x - 5y = 10$
(b) Sketch the graphs of each of these lines.

6. Find the equation of the line determined by the data in each of the following.
(a) y-intercept −1, slope $\frac{3}{4}$
(b) x-intercept 2, slope 5
(c) x-intercept 3, y-intercept 4
(d) x-intercept −3, y-intercept −8

7. Find the equation of the line with y-intercept 3 passing through the point $P(2, 4)$.

8. Write the defining equation of the family of lines:
(a) With slope $\frac{1}{5}$
(b) With y-intercept 17
(c) Parallel to the x-axis
(d) Passing through the origin

9. Determine the equation of a line through (9, 10) with slope $\frac{1}{11}$.

10. (a) Find the equation of the line $A(-1, 1)$, $B(6, 6)$.

Test your skill:
10% of 16

(b) Find the equation of the line $B(6, 6)$, $C(-8, -4)$.
(c) What can you conclude about A, B and C?

11. (a) Find the coordinates of the point of intersection of the x-axis and the line defined by each of the following.

(i) $2x + 5y = 10$	(ii) $3x - 2y = 6$
(iii) $3x + 4y - 12 = 0$	(iv) $x + y = 1$
(v) $2x + 4y = 9$	(vi) $3x + 5y + 15 = 0$

(b) State the x-intercepts of each of the lines.

12. Using the equations of question 11(a), find their points of intersection with the y-axis and hence the y-intercept of each line.

13. State the slope, x-intercept and y-intercept of the lines represented by each of the following.

(a) $3x - 2y + 6 = 0$	(b) $5x + 3y - 10 = 0$
(c) $2x - 3y + 2 = 0$	(d) $5x - 20 = 0$
(e) $9y - 27 = 0$	(f) $2x + 3y + 7 = 0$

3.4 PARALLEL AND PERPENDICULAR LINES

INVESTIGATION 3.4

1. (a) Referring to your results in INVESTIGATION 3.3, compare the slopes of parallel lines.
(b) If two lines L_1 and L_2 have slopes m_1 and m_2 respectively and if $L_1 \| L_2$, what must be true of m_1 and m_2?
(c) If two lines L_1 and L_2 have slopes m_1 and m_2 and if $m_1 = m_2$, then how are L_1 and L_2 related?

2. (a) Draw the graph of the line L_1 whose equation is $y = x$.
(b) What is the slope of the line $y = x$?
(c) Draw the graph of $y = x + 5$.
(d) What is the slope of the line $y = x + 5$?
(e) How are the two lines related?
(f) Through the origin draw a line, L_2, perpendicular to L_1, $y = x$.
(g) From the graph, determine the slope of L_2.
(h) How are the slopes of L_1 and L_2 related?
(i) Find the equation of L_2.

3. (a) Draw the graph of $y = 2x$.
(b) Through the origin draw a line perpendicular to the line $y = 2x$.
(c) What is the slope of $y = 2x$? Select another point on the line perpendicular to $y = 2x$ and calculate its slope.
(d) How are the slopes related?

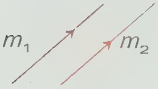
4. (a) State the slope and y-intercept of the line whose equation is $y = \frac{1}{4}x - 3$.
(b) Graph $y = \frac{1}{4}x - 3$.
(c) Draw a line perpendicular to $y = \frac{1}{4}x - 3$.
(d) What is the slope of the perpendicular line?

5. (a) From your results in questions 2, 3 and 4, what appears to be the relationship between the slopes of perpendicular lines?
(b) If two lines L_1 and L_2 have slopes m_1 and m_2 respectively and if

$L_1 \perp L_2$, what is the product $m_1 m_2$?
(c) Check your answer to (b) by using the results of questions 2, 3 and 4.

6. If two lines are found to have slopes whose product is -1, what type of lines are they?

Parallel lines



$m_1 = m_2$

Perpendicular lines

$m_1 \times m_2 = -1$

EXAMPLE 1. *Find the equation of the line through $(-4, 2)$ and perpendicular to the line $y = \frac{3}{4}x + 1$.*

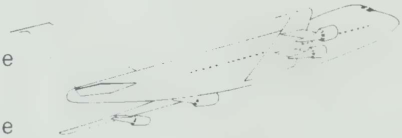
Solution The slope of $y = \frac{3}{4}x + 1$ is $\frac{3}{4}$. Since the lines are perpendicular, $m_1 \times m_2 = -1$.
$\therefore$ the slope of a line through $(-4, 2)$ is $-\frac{4}{3}$

$$y - y_1 = m(x - x_1)$$
$$y - 2 = -\tfrac{4}{3}(x + 4)$$
$$y - 2 = -\tfrac{4}{3}x - \tfrac{16}{3}$$
$$y = -\tfrac{4}{3}x - \tfrac{10}{3}$$
$$\text{or} \quad 4x + 3y + 10 = 0$$

EXERCISE 3-4

A 1. Find the slope of (i) a line parallel to and (ii) a line perpendicular to the following lines.

(a) $y = 2x + 3$ (b) $y = 4x - 1$ (c) $y = -3x + 1$
(d) $y = -x - 4$ (e) $y = \frac{1}{3}x + 4$ (f) $y = \frac{4}{3}x + 2$
(g) $y + 6x = 1$ (h) $y - 5x + 3 = 0$ (i) $2y + 3x = 4$
(j) $2x + 4y - 7 = 0$ (k) $3x - 2y = 7$ (l) $x - 3y + 1 = 0$



B 2. Find the equation of the line parallel to $y = 4x - 1$ and containing the point $(3, -2)$.

3. Find the equation of the line through $(7, 11)$ and perpendicular to the line $y = \frac{4}{3}x + 7$.

4. Find the equation of the line parallel to $2x + 3y = 7$ and containing the point $(3, 2)$.

5. Find the equation of the line through $(-2, -3)$ and perpendicular to the line $5x - 3y + 2 = 0$.

6. Find an equation of a line through $(2, -7)$ and
(a) Parallel to the x-axis. (b) Parallel to the y-axis.

7. Find the equation of a line passing through the origin perpendicular to $3x + 4y + 5 = 0$.

8. A line has y-intercept 3 and is perpendicular to the line $y = \frac{1}{5}x - 7$. Find an equation for the line.

9. Find an equation of the line with slope $\frac{5}{4}$ having the same x-intercept as the line $3x + 4y = 12$.

10. (a) Graph $A(2, -3)$, $B(5, 1)$, $C(0, -4)$, $D(5, 2)$.
(b) Find an equation of the line through A perpendicular to AB.
(c) Find an equation of the line through A parallel to BC.

11. Prove that the following points are the vertices of a right-angled triangle.
(a) $(-3, 2)$, $(7, 2)$, $(5, 6)$ (b) $(-6, 1)$, $(-2, -7)$, $(-4, -8)$

An airline flies 8 flights per day, 365 d/a from Toronto to Vancouver with 200 passengers per flight.
Find the annual cost of serving coffee on this run if coffee costs 20¢ per cup and each passenger has two cups.

12. (a) Prove that the following points are the vertices of a parallelogram.
(i) (2, 1), (14, 11), (6, 5), (−6, −5)
(ii) (6, 5), (2, 7), (−2, −1), (2, −3)
(b) What kind of parallelogram is (a) (ii)?

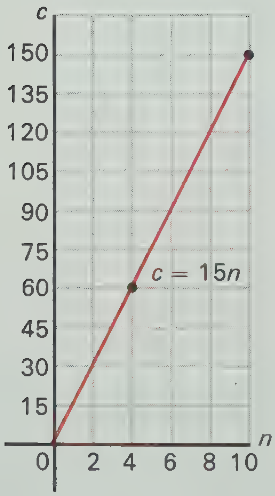
3.5 APPLICATIONS

The problems in this section give some applications of the linear equations we have been considering.

EXAMPLE 1. *(a) Taking the cost of gasoline to be 15¢/ℓ, make a table of values for the cost of 0 to 10 ℓ.*
(b) Illustrate the relationship between the number of litres and their cost graphically.

Solution (a)

No. litres	n	0	1	2	3	4	5	6	7	8	9	10
Total cost ¢	c	0	15	30	45	60	75	90	105	120	135	150



(b) (See figure.) Note that the graph obtained in Example I is a straight line. The equation of this line (the relationship between n and c) is $c = 15n$. Comparing this with the equation $y = mx$, what is the slope of the line in the graph?

The quotient $\frac{c}{n}$ is always 15. We say the cost, c, depends directly on the number of litres, n. c varies directly as n or c is in **direct proportion** to n.

EXAMPLE 2. *The circumference of a circle is directly proportional to its diameter. If a circle of diameter 10* cm *has a circumference of 31* cm:
(a) calculate the circumference of a circle of diameter 14 cm.
(b) calculate the radius of a circle of circumference 47.2 cm.

Solution Let the circumference in centimetres be c and the diameter in centimetres be d. Because c is directly proportional to d we say $\frac{c}{d} =$ a constant. To find the value of the constant we know that if $d = 10$, then $c = 31$. Thus $\frac{c}{d} = \frac{31}{10} = 3.1$. The relationship between c and d is $c = 3.1d$.

(a) If $d = 14$, $c = 3.1 \times 14 = 43.4$. Therefore the circumference is 43.4 cm.

(b) If $c = 47.2$, $47.2 = 3.1d$. Therefore $d = \frac{47.2}{3.1} = 15.2$ and the radius is 7.6 cm (correct to two figures).

EXERCISE 3-5

Addition:
HOCUS
POCUS
PRESTO

B **1.** In a given electrical circuit the voltage is directly proportional to the current. If the voltage is 9 V, then the current is 12 A.
(a) Find an equation representing the relation between the voltage (V) and the current (I).
(b) By comparing this equation to $y = mx$, where y corresponds to V and x corresponds to I, find the slope of the line.
(c) Draw the graph of the relation in (a) using the slope and the co-ordinates of one point on the line.
(d) From the graph complete the following table.

V	I
	8 A
3 V	
	2 A
4.5 V	

2. (a) The distance d, in kilometres, which a car travels at a uniform speed is directly proportional to the time taken, t, in hours. If the car travels 300 km in 6 h, find an equation representing the relationship between distance and time.
(b) Draw the graph of this relation.
(c) From the graph find:
 (i) the distance travelled in 2.5 h.
 (ii) the time required to travel 500 km.

3. (a) Simple interest i, in dollars, is earned in direct proportion to the time t, in years. If \$35 interest is earned in 5 a, how much interest is earned in 2 a?
(b) How long would it take to earn \$3.50?

a
is the short form of annum, the Latin word for year.

4. The length of the diagonal of a square is directly proportional to the length of the side of the square. If a square has a side of 10 m and a diagonal of approximately 14 m, find the length of a diagonal of a 50 m square.

5. The volume of a certain gas, which is kept at a constant pressure, is directly proportional to the temperature in kelvin. If 625 cm^3 at 500 K are raised 50 K, what will be the resulting volume?

$0° C = 273.15$ K

6. The resistance in an electrical circuit is proportional to the length of the wire. 1200 m of this wire has a resistance of 25 Ω. Draw a graph and determine from the graph the resistance of 3 km of this wire.

1 Ω = one ohm

7. The length of a shadow is proportional to the height of an object. A Boy Scout found that a 6 m pole casts a 2.4 m shadow. How high would a telephone pole be if its shadow was 9.7 m long?

The previous exercises have been examples of direct proportion in which one variable, y, is directly proportional to another, x; that is, $y = mx$ for some constant m. A second type of example follows.

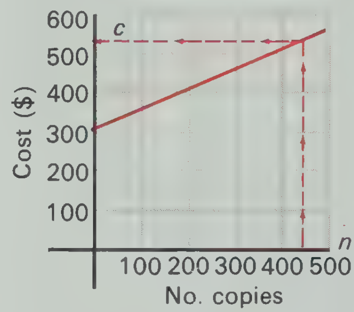
The cost c of printing a school yearbook is partly a constant sum (the cost of setting the type, etc.) and partly variable (directly proportional to

the number of copies). Thus the total cost is the sum of a constant, b, and a direct proportion. In this example $c = b + mx$, where b is the constant fixed cost, m the proportionality constant, x the number of copies, and c the total cost.

EXAMPLE 3. *The cost of a school yearbook will be \$300 plus 50¢ per copy.*
(a) Write an equation to represent the relationship between c and n.
(b) Draw a graph.
(c) With respect to the graph, what does 0.50 represent? What does 300 represent?
(d) From the graph, how much will 450 yearbooks cost?

Solution

(a) $c = 300 + 0.50n$, or
$c = 0.50n + 300$



(c) By comparing $c = 0.50n + 300$ to $y = mx + b$, it can be seen that 0.50 represents m, the slope of the line, and 300 represents b, the y-intercept of the line.
(d) From the graph, 450 yearbooks will cost \$525.

8. The cost of having a party will be \$5 plus \$1 for each guest who attends.
(a) Express the relationship between the total cost, c, and the number of guests, n.
(b) Draw the graph of this linear relation using the slope and the co-ordinates of one point on the graph.
(c) Find from the graph the cost of having a class party of 15 girls and 13 boys.

Test your skill:
3% of \$8.00

9. The cost of a certain entertainment can be determined two ways: (i) \$1.75 per guest; (ii) \$1.00 per guest plus the fixed cost of \$12.00 to rent a hall.
(a) By plotting the number of guests along the x-axis and the total cost along the y-axis, draw a graph to represent methods (i) and (ii) on the same set of axes.
(b) For how many guests is the total cost the same whether you use methods (i) or (ii)?
(c) What is this cost?
(d) For how many guests is it preferable to calculate the cost by method (i)?
(e) For how many guests is it preferable to calculate the cost by method (ii)?

10. The yearly expenses for operating an automobile are partly constant

(the depreciation) and partly dependent upon the number of kilometres travelled. A certain car costs \$0.05/km to operate and depreciates \$800 during its first year. How much will it cost to drive this car 10 000 km?

We close this chapter by considering an application which does not lead to a straight line graph. If $\frac{y}{x}$ is a constant, y is directly proportional to x.

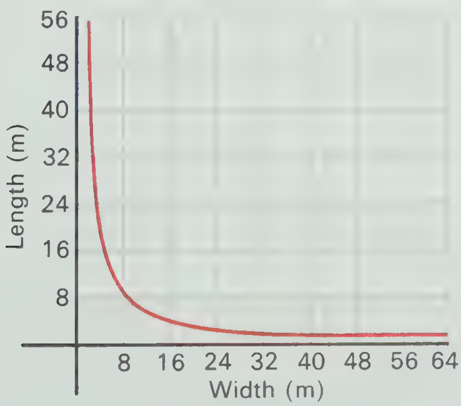
If yx is a constant we say y is inversely proportional to x.

EXAMPLE 4. *If a rectangle has a constant area its length, l, is inversely proportional to its width, w.*
(a) If the area of a certain rectangle is 64 m^2*, complete the following table.*

l (m)	1	2	4	8		32	64
w (m)		32			4		

(b) Use the values in the table to draw a graph.
(c) Find l when $w = 12$ m.

Solution
(a) We leave this to the student.
(b)



(c) From the graph, when $w = 12$ m, l appears to be about 5.5 m.

11. The current, I in amperes, in an electrical circuit is inversely proportional to the resistance, R. If the current is 5 A, then the resistance is 2 Ω.
(a) Find an equation representing the relation between the current, I, and the resistance, R.

(b) Complete the following table.

R Ω	1	2	4	5	6	8	10
I A							

(c) Draw a graph.
(d) From the graph, complete the following.

R	$3\frac{1}{2}$		
I		$6\frac{1}{2}$	$3\frac{1}{2}$

(e) Draw a second graph by plotting R and $\frac{1}{I}$.

(f) What kind of graph is this?

12. The time taken to travel a fixed distance is inversely proportional to the speed. Suppose you are planning a 300 km trip.
(a) Draw up a table of values showing the relationship between time and speed to make this trip.
(b) Draw a graph.
(c) From the graph, how long will it take if an average speed of 45 km/h can be maintained?

1 kPa = one kilopascal

13. When the temperature is kept constant, the volume of a gas is inversely proportional to the pressure. If 300 cm^3 of gas are under 315 kPa of pressure, what would the pressure become if the volume were increased to 360 cm^3?

REVIEW EXERCISE

B

1. Determine the slope of the line passing through:
(a) (5, 10) and (8, 6)
(b) (16, 1) and (10, −2)
(c) (2, 1) and (5, −6)
(d) (−4, −2) and (−1, 7)
(e) (−1, 4) and (−3, −5)
(f) (6, −5) and (2, −7)

2. (a) The line joining $(x, 2)$ and (1, 0) is parallel to the line joining (−2, 1) and (2, 3). Find x.
(b) Repeat (a) substituting the word "perpendicular" for the word "parallel".

3. Plot the points $G(4, 1)$, $H(2, 3)$, $J(6, 4)$, $K(8, 2)$.
(a) Show that $GHJK$ is a parallelogram.
(b) Is $GHJK$ a rectangle?

4. Determine the equation of each line in the form $Ax + By + C = 0$.
(a) Passing through (−4, 3) with slope $\frac{1}{2}$
(b) Passing through (0, 5) with slope −2

(c) Passing through (2, 0) with slope $\frac{3}{4}$
(d) Passing through $(-1, -3)$ with slope $-\frac{2}{3}$
(e) Passing through $(2, -5)$ with slope $\frac{2}{5}$

5. Write the equation of the line with the same y-intercept as $y = 3x + 4$ and having slope -1.

6. Write the equation of the line perpendicular to $y = 3x - 2$ and passing through (3, 2).

7. Determine the equation of a line through (1, 2) with slope $-\frac{1}{4}$.

8. Determine the equation of the line through $(-2, -3)$ parallel to $5x + 3y = 2$.

9. Determine the equation of the line passing through (3, 5) and perpendicular to $3x - 2y + 13 = 0$.

10. Write the equation of the line whose x- and y-intercepts are 5 and -3 respectively.

11. Find the slope and y-intercept of $3x + 2y = 7$.

12. Determine the value of k so that:
(a) $4x - ky = 7$ has slope 3.
(b) $kx - y = -15$ has x-intercept 5.

13. What must be the value of m if the line $y = mx + 7$ passes through $(-2, 5)$?

14. Find the equation of the straight line parallel to $6x - 7y = 13$ and passing through $(-2, -5)$.

15. Find the equation of the straight line through (2, 6) and perpendicular to the line through $(-3, 5)$ and $(7, -1)$.

C **16.** Find the point on the y-axis which is on the line through $(6, -3)$ and $(-12, 12)$.

17. A perpendicular from the point $(4, -2)$ to a line meets the line at the point $(-3, 1)$. Find the equation of the line.

18. Find the equation of a line with slope $\frac{2}{3}$ and having the same y-intercept as the line $2x - 5y + 20 = 0$.

19. Find the coordinates of the point on the x-axis so that the slope of the line joining it to $(5, -2)$ is $-\frac{3}{4}$.

20. The y-intercept of a line is -4 and it passes through $(-2, 5)$. Find the equation of the line.

21. Find the value of k if $3x - 2y + 5 = 0$ and $6x + ky + 9 = 0$ are parallel.

REVIEW AND PREVIEW TO CHAPTER 4

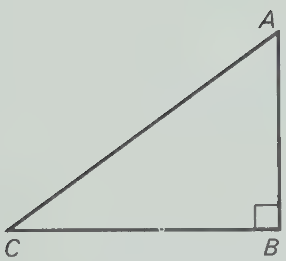
EXERCISE 1 *Percent*

1. Evaluate the following.

(a) 10% of 50	(b) 25% of 300	(c) 60% of 80
(d) 5% of 20	(e) 21% of 400	(f) 7% of 63
(g) 2% of 65	(h) 5.5% of 30	(i) 10.5% of 200
(j) 12.5% of 70	(k) 2.5% of 35	(l) 0.5% of 10
(m) 2.25% of 60	(n) 3.75% of 44	(o) 7.2% of 33
(p) 100% of 71	(q) 200% of 120	(r) 500% of 6

2. (a) What percent of 30 is 15?
(b) What percent of 20 is 5?
(c) What percent of 60 is 15?
(d) What percent of 70 is 7?
(e) What percent of 2.5 is 0.5?
(f) What percent of 3.6 is 0.06?
(g) What percent of 50 is 150?
(h) What percent of 60 is 300?

EXERCISE 2 *Pythagorean Relationship*



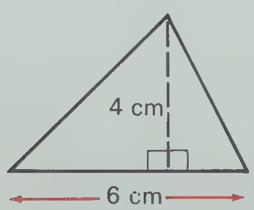
$(AC)^2 = (AB)^2 + (BC)^2$

1. Use the Pythagorean relationship to calculate the missing lengths in $\triangle ABC$.
(a) $\angle B = 90°$, $AB = 4$ cm, $BC = 3$ cm
(b) $\angle B = 90°$, $AB = 12$ cm, $BC = 5$ cm
(c) $\angle A = 90°$, $AB = 6$ cm, $AC = 4$ cm
(d) $\angle C = 90°$, $AB = 10$ cm, $BC = 3$ cm
(e) $\angle A = 90°$, $BC = 12$ cm, $AC = 7$ cm
(f) $\angle C = 90°$, $AB = 20$ cm, $BC = 16$ cm

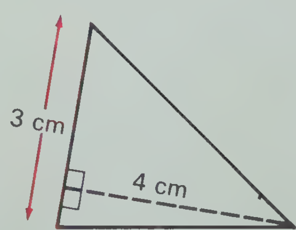
EXERCISE 3 *Area of a Triangle*

1. Calculate the areas of the following triangles (where possible).

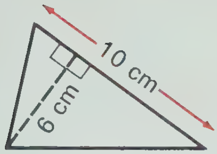
(a)



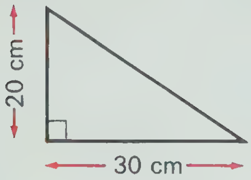
(b)



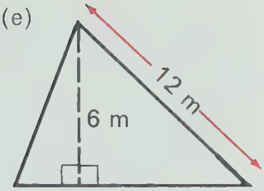
(c)

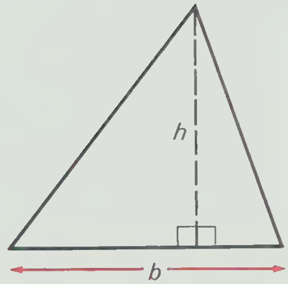


(d)



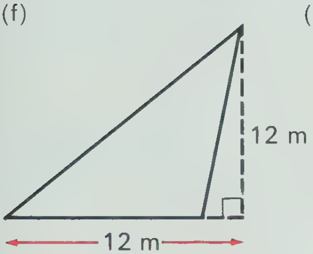
(e)



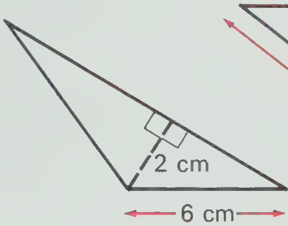


$A = \frac{1}{2} \times b \times h$

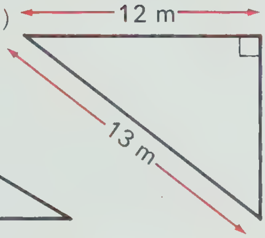
(f)



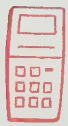
(g)



(h)



$$m = \frac{\triangle y}{\triangle x} = \frac{y_2 - y_1}{x_2 - x_1}$$



Determine the slope of the lines passing through:

1. (0.255, 3.215) and (1.271, 5.327)
2. (3.217, 4.615) and (8.575, 2.125)
3. (4.385, 6.124) and (−3.255, −4.173)
4. (0.0255, −0.3125) and (−0.2557, 0.4255)
5. (−12.37, −4.21) and (5.72, −8.63)
6. (27.38, −33.81) and (16.81, −35.46)
7. (−0.713, 0.848) and (−0.991, −0.322)
8. (123.6, 47.8) and (−156.9, −66.8)
9. (−4.317, −5.481) and (−7.922, −10.473)
10. (−728.4, 561.2) and (555.5, −685.7)

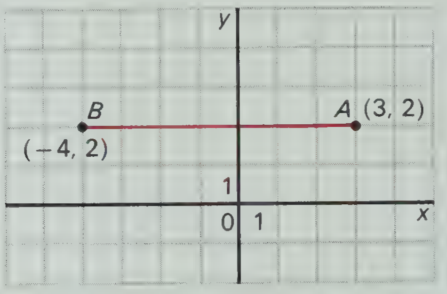
CHAPTER 4

Distance Between Two Points

In Chapter 3 we studied one of the important properties of a straight line or straight line segment—*slope*. In this chapter we are going to be concerned with another characteristic of a line segment, its *length*.

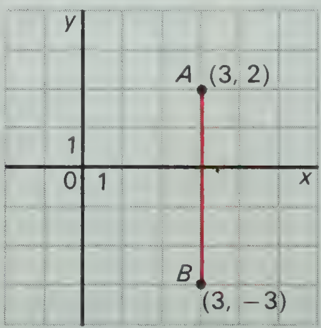
4.1 LENGTH OF A LINE SEGMENT

The length of a horizontal line segment may be determined by calculating $|\triangle x|$ for the line.



The length of AB, $l_{AB} = |\triangle x|$
$= |(3) - (-4)|$
$= 7$

The length of a vertical line segment may be determined by calculating $|\triangle y|$ for the line.

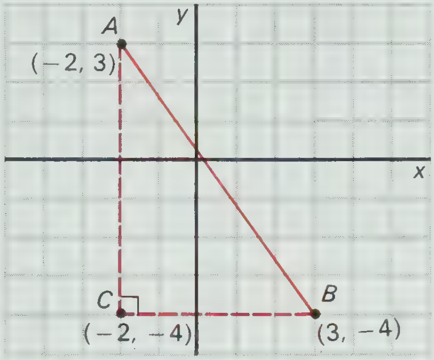


The length of AB; $l_{AB} = |\triangle y|$
$= |(2) - (-3)|$
$= 5$

The length of any other line segment may be found using the Pythagorean theorem.

EXAMPLE 1. *Find the length of the line segment from $A(-2, 3)$ to $B(3, -4)$.*

Solution Construct right triangle ABC.



For the line segment AB,

$$\text{length of } BC = |\triangle x| = |(3) - (-2)| = 5$$

$$\text{length of } AC = |\triangle y| = |(3) - (-4)| = 7$$

By Pythagoras, $(AB)^2 = (BC)^2 + (AC)^2$

$$\therefore AB = \sqrt{(BC)^2 + (AC)^2} = \sqrt{5^2 + 7^2} = \sqrt{25 + 49} = \sqrt{74}$$

Find five sets of whole numbers a, b, and c such that $a^2 + b^2 = c^2$.

Since $(BC)^2 = \triangle x^2$ and $(AC)^2 = \triangle y^2$ then $l_{AB} = \sqrt{\triangle x^2 + \triangle y^2}$

EXAMPLE 2. *Find the length of the line segment joining $C(1, -3)$ to $D(4, 7)$.*

Solution

$$l_{CD} = \sqrt{\triangle x^2 + \triangle y^2} = \sqrt{(-3)^2 + (-10)^2} = \sqrt{9 + 100} = \sqrt{109}$$

EXERCISE 4-1

A

1. Determine the lengths of the following line segments.

(a) $A(2, 4)$, $B(2, 2)$
(b) $M(8, 6)$, $N(2, 6)$
(c) $D(-3, 4)$, $E(-3, 8)$
(d) $F(2, -1)$, $G(-3, -1)$
(e) $M(0, 7)$, $N(-4, 7)$
(f) $T(4, 0)$, $S(-6, 0)$
(g) $R(6, -5)$, $P(2, -5)$
(h) $Q(-6, -7)$, $T(-6, -13)$
(i) $L(2, -3)$, $S(2, 5)$
(j) $K(3, -11)$, $T(-5, -11)$

B 2. Calculate the lengths of the following line segments.

(a) $A(2, 3)$, $B(1, 1)$
(b) $C(4, 11)$, $B(2, 6)$
(c) $X(3, -7)$, $P(2, -6)$
(d) $Q(-4, -2)$, $S(3, -4)$
(e) $M(4, 8)$, $N(4, 2)$
(f) $S(3, 6)$, $T(-4, 6)$
(g) $O(0, 0)$, $M(-4, 6)$
(h) $F(-2, -4)$, $G(-3, -8)$
(i) $J(-2, -6)$, $T(2, -9)$
(j) $A(0, -3)$, $B(-2, -6)$
(k) $D(0, 5)$, $F(4, 0)$
(l) $S(5, -5)$, $M(-5, 5)$

3. Find the lengths of the sides of a triangle with vertices $A(10, 0)$, $B(9, 7)$, $C(1, 5)$.

4. Find the perimeter of a triangle whose vertices are $D(5, 7)$, $E(1, 10)$, $F(-3, -8)$.

5. In a circle, the diameter has endpoints $(4, 2)$ and $(-2, 4)$. Find its length.

6. Determine the type of the following triangles by finding the lengths of the sides in each case.

(a) $(3, 2)$, $(6, -5)$, $(10, 5)$
(b) $(3, -2)$, $(0, 2)$, $(4, 1)$
(c) $(10, 0)$, $(9, 7)$, $(1, 5)$
(d) $(-1, 0)$, $(1, 0)$, $(0, \sqrt{3})$

7. A quadrilateral has vertices $P(-3, 4)$, $Q(10, 7)$, $R(2, -8)$ and $S(-5, -1)$. What are the lengths of the diagonals?

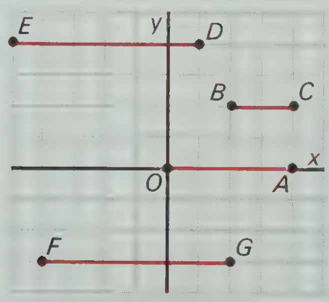
C 8. Show that $(2, 3)$ is the midpoint of the line segment joining $(7, 8)$ and $(-3, -2)$.

9. Find the coordinates of a point on the x-axis which is equidistant from the points $A(4, 8)$ and $(B(6, 6)$.

10. Find the coordinates of a point on the y-axis which is equidistant from $C(7, -1)$ and $D(1, 3)$.

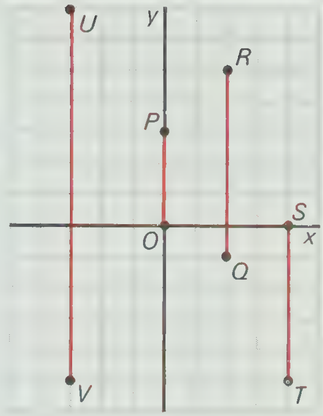
4.2 MIDPOINT OF A LINE SEGMENT

INVESTIGATION 4.2



1. (a) From the figure, find by inspection the coordinates of the midpoints of OA, BC, DE, and FG.
(b) How is the x-coordinate of the midpoint related to the x-coordinates of the endpoints in each case?
(c) Suppose the midpoint of FG is called H. Using the length formula find FH and HG to check that $FH = HG$.

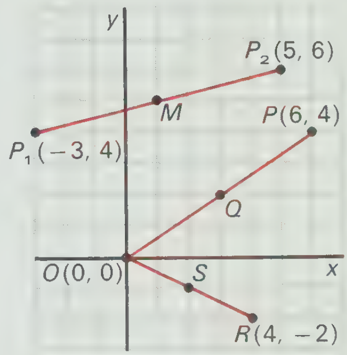
2. (a) From the figure, find by inspection the coordinates of the midpoints of OP, QR, ST, and UV.
(b) How is the y-coordinate of the midpoint related to the y-coordinates of the endpoints?
(c) By calling W the midpoint of UV, show that $UW = WV$.



3. (a) From the figure, find the coordinates of the midpoints Q, S, and M.
(b) Find the lengths of P_1M, MP_2 and P_1P_2 and check that

$$P_1M = MP_2 = \tfrac{1}{2}P_1P_2.$$

(c) How is the x-coordinate of M related to the x-coordinates of P_1 and P_2?
(d) How is the y-coordinate of M related to the y-coordinates of P_1 and P_2?



> The coordinates of the midpoint of the segment joining $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$ are $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

EXAMPLE 1. *Find the midpoint of the segment joining $A(-3, 5)$ and $B(-7, -6)$.*

Solution

$$x = \frac{x_1 + x_2}{2} = \frac{(-3) + (-7)}{2} = \frac{-10}{2} = -5 \qquad y = \frac{y_1 + y_2}{2} = \frac{5 + (-6)}{2} = -\tfrac{1}{2}$$

The midpoint of AB is $(-5, -\tfrac{1}{2})$.

Starting with the word "tool" and changing one letter at a time to form a new word, can you reach "chin" in four changes?

tool
1. —
2. —
3. —
4. chin

EXERCISE 4-2

B 1. Find the midpoint of the line segments joining the following pairs of points.

(a) $(5, 6)$ and $(7, 8)$ (b) $(4, 0)$ and $(6, 10)$
(c) $(-3, 6)$ and $(1, 2)$ (d) $(6, -3)$ and $(5, 4)$
(e) $(9, -3)$ and $(-5, 2)$ (f) $(-4, -6)$ and $(-3, -8)$
(g) $(-6, 9)$ and $(5, -13)$ (h) $(0, 0)$ and $(-4, -3)$

(i) $\left(-\frac{7}{3}, -\frac{1}{4}\right)$ and $\left(\frac{4}{3}, \frac{5}{4}\right)$ (j) $\left(-\frac{1}{2}, -\frac{3}{5}\right)$ and $\left(\frac{3}{2}, -\frac{8}{5}\right)$

2. $A(-4, 4)$, $B(2, -2)$, and $C(6, 2)$ are the vertices of a triangle.
(a) Find the coordinates of M(the midpoint of AC) and Q(the midpoint of AB).
(b) Compare the slopes of QM and BC.
(c) How are the lines QM and BC related?
(d) Find the length of QM and of BC showing that the length of a line segment joining the midpoints of two sides of a triangle is equal to one half the third side.

3. $X(-2, 1)$, $Y(-6, -1)$, and $Z(2, -7)$ are the vertices of a triangle.
(a) Compare the slopes of XY and XZ.
(b) How are the lines XY and XZ related?
(c) What kind of triangle is $\triangle XYZ$?
(d) Find the coordinates of W, the midpoint of YZ.
(e) Compare the lengths of XW, YW and ZW.

4. What is the centre of a circle whose diameter has endpoints $(4, 2)$ and $(-2, 4)$?

C 5. The midpoint of PQ is the origin. P has coordinates $(4, 5)$. What are the coordinates of Q?

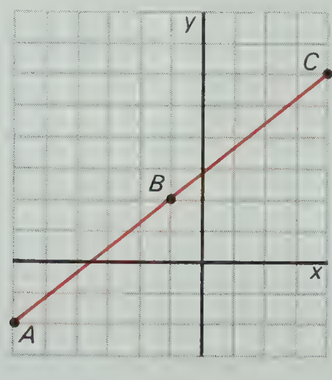
6. The vertices of a parallelogram are $(2, 1)$, $(7, 2)$, $(8, 5)$, and $(3, 4)$. Find the midpoint of each diagonal.

7. $(-1, 6)$ is the midpoint of a line segment. If $(2, 8)$ is one endpoint, what is the other endpoint?

4.3 COLLINEAR POINTS

Collinear points are points which lie on the same straight line.

INVESTIGATION 4.3



Consider three points $A(-6, -2)$, $B(-1, 2)$, and $C(4, 6)$, as illustrated here.

1. (a) Compare slope AB, slope BC, and slope CA.
(b) What can we conclude about A, B, and C?
(c) How many slopes are necessary to show that A, B, and C are collinear?

2. (a) Find the lengths of AB, BC, and CA.
(b) What relationship exists among these lengths?
(c) Draw a conclusion.

3. (a) Using the slope of AB found in 1(a) and the slope point form of the equation of a straight line found in Chapter 2, find the equation of AB.
(b) For C to lie on the line AB, the coordinates of C must satisfy the equation of AB. Check to see if C is on AB.

4. (a) Find the equation of BC.
(b) Compare the equation of BC with the equation of AB and draw a conclusion regarding A, B, and C.

> Tests for Collinearity
> Three points, A, B, and C are collinear if one of the following can be proved.
> 1. Slope AB = slope BC.
> 2. $AB + BC = AC$, if B lies between A and C.
> 3. The coordinates of C satisfy the equation of AB.
> 4. The equation of AB is the same as the equation of BC.

EXAMPLE 1. *Determine whether or not the points $A(-7, -2)$, $B(-2, 1)$, and $C(3, 4)$ are collinear.*

Solution 1 BY SLOPES

$$m_{AB} = \frac{\triangle y}{\triangle x} = \frac{1-(-2)}{-2-(-7)} = \frac{3}{5} \qquad m_{BC} = \frac{\triangle y}{\triangle x} = \frac{4-1}{3-(-2)} = \frac{3}{5}$$

$$m_{AB} = m_{BC}$$

$\therefore$ A, B, and C are collinear.

Addition:
MOON
MEN
CAN
REACH

Solution 2 BY LENGTHS

$$l_{AB} = \sqrt{\triangle x^2 + \triangle y^2} = \sqrt{5^2+3^2} = \sqrt{25+9} = \sqrt{34}$$

$$l_{BC} = \sqrt{\triangle x^2 + \triangle y^2} = \sqrt{5^2+3^2} = \sqrt{25+9} = \sqrt{34}$$

$$l_{AC} = \sqrt{\triangle x^2 + \triangle y^2} = \sqrt{10^2+6^2} = \sqrt{100+36} = \sqrt{136} = 2\sqrt{34}$$

$\therefore AB + BC = AC$
$\therefore$ A, B, and C are collinear.

EXERCISE 4-3

B

1. Determine whether the following points are collinear.
(a) $A(1, -1)$, $B(0, 1)$, $C(-1, 3)$
(b) $D(3, 0)$, $E(0, -3)$, $F(1, -4)$
(c) $G(2, 8)$, $H(-4, 8)$, $J(10, 8)$
(d) $M(-2, 7)$, $R(0, 2)$, $T(5, -10)$
(e) $G(1, 1)$, $S(-2, -5)$, $Q(3, 5)$
(f) $A(3, 2)$, $K(-1, -2)$, $L(-4, -6)$

2. (a) Find k if $(1, 2)$, $(0, 5)$, and $(3, k)$ are collinear.
(b) Find d if $(1, -2)$, $(-4, 3)$, and $(d, -3)$ are collinear.

3. Prove that $A(1, 7)$, $B(3, -1)$, and $C(4, -5)$ are collinear by comparing:
(a) slopes.
(b) lengths.

4. Prove that $D(-10, -3)$, $E(2, 0)$, and $F(6, 1)$ are collinear, using an equation method.

5. A triangle has vertices $A(3, 3)$, $B(5, -1)$, and $C(9, 5)$.
(a) Find the midpoints D, E, and F of BC, CA, and AB respectively.
(b) Find the equations of the medians of $\triangle ABC$.
(c) Solve the equations of two of the medians and show that the point of intersection lies on the third median.
(d) The point of intersection of the three medians of a triangle is called the **centroid**. What is the centroid, G, of $\triangle ABC$?
(e) A **perpendicular bisector** of a line segment is a line perpendicular to the given line segment passing through its midpoint. By first finding the slopes of BC, CA, and AB, determine the slopes of the perpendicular bisectors through D, E, and F.
(f) Find the equations of the perpendicular bisectors of $\triangle ABC$.
(g) The point of intersection of the three perpendicular bisectors of a triangle is called the **circumcentre**. What is the circumcentre, R, of $\triangle ABC$?
(h) Compare the lengths of RA, RB, and RC.
(i) A line drawn through a vertex of a triangle perpendicular to the opposite side is called an **altitude**. Using the slopes found in (e) and the coordinates of the given vertices, find the equations of the three altitudes.
(j) The three altitudes of a triangle intersect at the **orthocentre**. Find the orthocentre, O, of $\triangle ABC$.
(k) Use the method of your choice to show that the centroid G [found in (d) above], the circumcentre R [found in (g)], and the orthocentre O [found in (j)] are collinear.
(l) Finally, compare the lengths of GR and GO.

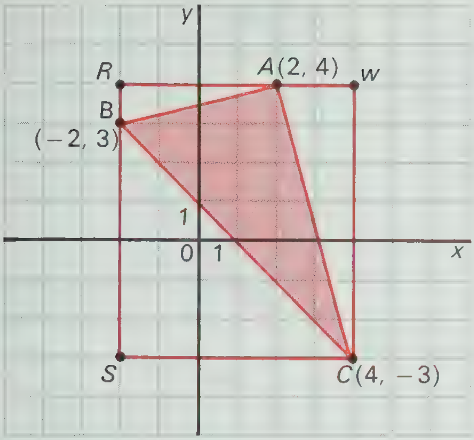
407

$4^3 + 0^3 + 7^3$
$= 64 + 0 + 343$
$= 407$
Find three other three-digit numbers like this.

4.4 AREA OF A TRIANGLE

Given the coordinates of the vertices of a triangle, the area of the triangle may be determined as follows.

EXAMPLE 1. *Determine the area of $\triangle ABC$ with vertices $A(2, 4)$, $B(-2, 3)$, $C(4, -3)$.*



Solution

Construct rectangle *RSCW*.
Area of rectangle *RSCW*
$= 6 \times 7$
$= 42$ square units.

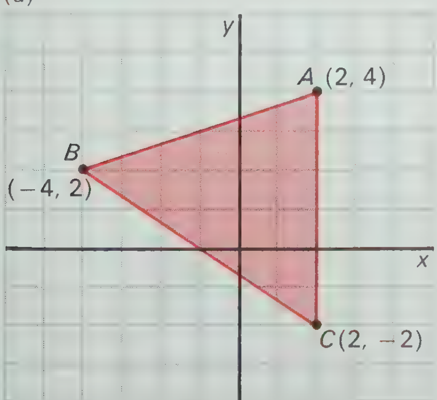
By subtracting the areas of $\triangle RBA$, $\triangle AWC$, and $\triangle BSC$ from the area of the rectangle, the area of $\triangle ABC$ will be found.

$$
\begin{aligned}
\triangle RBA &= \tfrac{1}{2} \times b \times h \\
&= \tfrac{1}{2} \times 1 \times 4 \\
&= 2 \text{ square units} \\
\triangle AWC &= \tfrac{1}{2} \times 2 \times 7 \\
&= 7 \text{ square units} \\
\triangle BSC &= \tfrac{1}{2} \times 6 \times 6 \\
&= 18 \text{ square units} \\
\triangle RBA + \triangle AWC + \triangle BSC &= 2 + 7 + 18 \\
&= 27 \text{ square units} \\
\therefore \text{Area of } \triangle ABC &= (\text{Area of } RSCW) - (27) \\
&= 42 - 27 \\
&= 15 \text{ square units}
\end{aligned}
$$

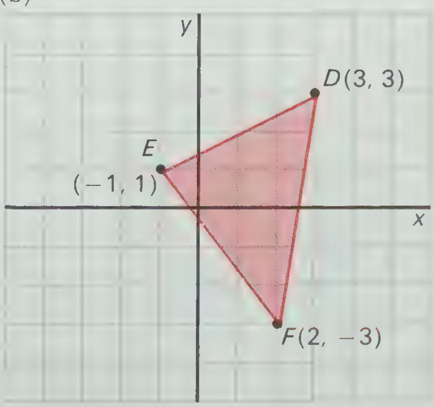
EXERCISE 4-4

1. Calculate the areas of the following triangles.

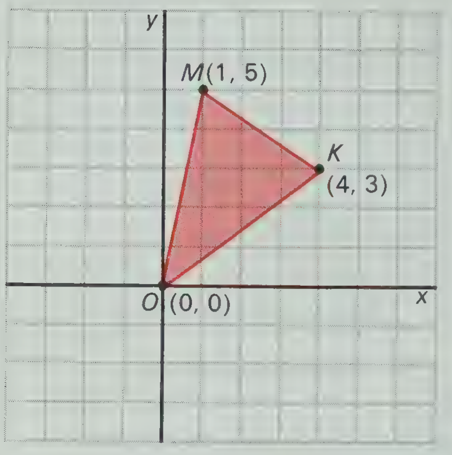
(a)



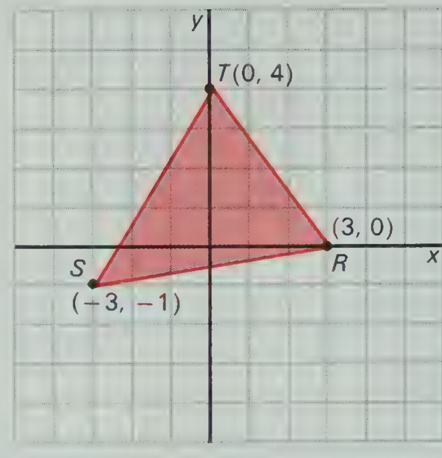
(b)



(c)



(d)

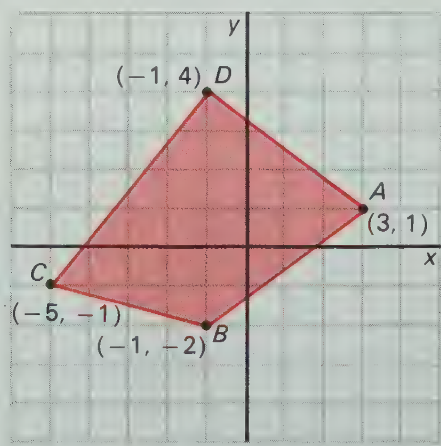


2. Calculate the areas of the following triangles.
(a) $A(4, 2)$, $B(6, 4)$, $C(2, 6)$
(b) $D(6, 3)$, $E(3, 2)$, $F(1, 7)$
(c) $G(3, 5)$, $H(6, 0)$, $J(1, 2)$
(d) $M(-4, 2)$, $N(3, -6)$, $T(-4, -3)$
(e) $P(2, -3)$, $L(-4, -5)$, $S(-3, -6)$
(f) $K(-5, -2)$, $R(-5, -6)$, $Q(8, 3)$
(g) $S(0, 4)$, $A(-2, 0)$, $B(3, 3)$
(h) $E(0, 0)$, $F(-2, -3)$, $H(-1, 6)$

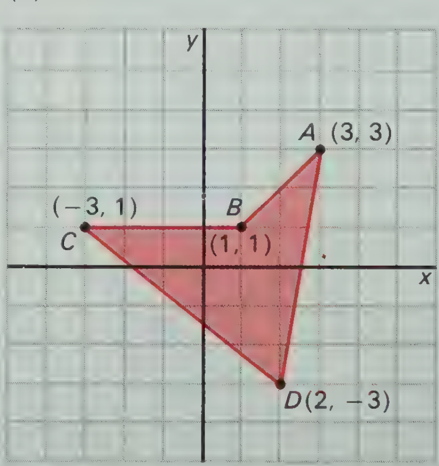
C 3. Triangle ABC has vertices $A(4, 6)$, $B(-3, 2)$, and $C(7, -2)$. Determine the coordinates of D, the midpoint of BC. AD is a median of $\triangle ABC$. Calculate the areas of $\triangle ABD$ and $\triangle ACD$. What conclusions can you draw from your results?

4. Calculate the areas of the following figures.

(a)



(b)



REVIEW EXERCISE

B **1.** Determine the lengths of the following line segments.
(a) $D(0, 3)$, $E(4, 2)$ (b) $A(-1, -1)$, $B(6, -3)$
(c) $G(9, 8)$, $H(3, 4)$ (d) $D(-1, -2)$, $E(3, -2)$
(e) $R(4, -3)$, $T(-2, 4)$ (f) $M(6, 6)$, $N(2, 2)$
(g) $Q(-3, 0)$, $R(0, -3)$ (h) $A(3, 0)$, $B(9, 0)$

2. Find the coordinates of the midpoint of the following line segments.
(a) $A(3, 6)$, $B(5, 8)$ (b) $C(-4, 6)$, $D(-8, -2)$
(c) $E(-1, 3)$, $F(7, -5)$ (d) $G(3, -8)$, $H(4, 5)$
(e) $I(-4, 11)$, $J(5, -2)$ (f) $K(-3, -5)$, $L(-7, -11)$

3. Determine whether the following sets of points are collinear.
(a) $A(-2, -2)$, $B(1, 1)$, $C(4, 4)$
(b) $D(3, 3)$, $C(4, 5)$, $E(7, 7)$
(c) $G(1, 3)$, $F(2, 5)$, $H(3, 7)$
(d) $R(-1, 4)$, $S(2, -6)$, $T(0, 1)$
(e) $M(1, 1)$, $N(3, 0)$, $R(5, -2)$

4. Calculate the areas of the following triangles.
(a) $A(5, 1)$, $B(3, 9)$, $C(1, 3)$
(b) $D(2, -4)$, $E(-4, 6)$, $F(-2, -8)$
(c) $G(3, 0)$, $D(-1, 5)$, $K(1, -7)$

5. (a) Find the area of the triangle $K(-3, 0)$, $L(5, 4)$, $M(9, 6)$.
(b) What kind of points are K, L, M?
(c) State five methods of showing points to be collinear.

6. By finding the lengths of the sides of the triangle with vertices $A(6, -2)$, $B(-2, 4)$, and $C(5, 5)$, determine what kind of triangle it is.

7. If $(4, -5)$ is the midpoint of a line segment joining $P(-1, 2)$ to Q, what are the coordinates of Q?

8. (a) Find the coordinates of the midpoints of the sides of the triangle whose vertices are $A(4, 6)$, $B(2, 1)$, and $C(6, 1)$.
(b) Find the area of $\triangle ABC$.
(c) Find the area of the triangle formed by joining the midpoints of the sides of $\triangle ABC$.
(d) How do the areas compare?

9. Find the area of quadrilateral $Q(6, 9)$, $R(0, 12)$, $S(-3, 6)$, $T(3, 3)$.

10. (a) Plot any four points on a piece of graph paper and label them A, B, C, and D. Join the points to form a quadrilateral.
(b) Find X, the midpoint of AB; Y, the midpoint of BC; Z, the midpoint of CD; and W, the midpoint of DA.
(c) Compare the lengths of XY and ZW.
(d) Compare the slopes of XY and ZW.
(e) What kind of figure is $XYZW$?

11. The sides of a triangle are represented by $3x + y + 4 = 0$, $x - y = 4$ and $x - 5y - 4 = 0$. Find:
(a) The vertices of the triangle.
(b) The length of the sides.
(c) The area of the triangle.
(d) The midpoints of the sides.

(e) The equations of the medians.
(f) The coordinates of the centroid.
(g) The slopes of the sides of the triangle.
(h) The equations of the altitudes.
(i) The coordinates of the orthocentre.

12. Prove that $A(7, 2)$, $B(-3, 4)$, and $C(2, 3)$ are collinear.

13. Calculate the length of $A(7, 2)$, $B(-3, 4)$ correct to two decimal places.

14. Find the area of a quadrilateral whose vertices are $P(4, -3)$, $Q(2, 3)$, $R(3, 3)$, and $S(-6, -1)$.

REVIEW AND PREVIEW TO CHAPTER 5

EXERCISE 1

Evaluate correct to 3 significant figures.

1. (a) 26.5×3.7 (b) 4.8×11.6
(c) 33.5×14.2 (d) 0.7×25.5
(e) 0.3×0.52 (f) 136×9.8
(g) 763×2.1 (h) 0.062×0.59

2. (a) $12.3 \times 7.4 \times 16.2$ (b) $66.5 \times 0.42 \times 1.1$
(c) $23.6 \times 13.3 \times 0.72$ (d) $798 \times 0.43 \times 1.07$
(e) $7.7 \times 0.005 \times 60$ (f) $605 \times 7.9 \times 0.35$

3. (a) $66.5 \div 32$ (b) $127 \div 2.7$ (c) $33.5 \div 8.6$
(d) $0.81 \div 3.2$ (e) $0.073 \div 0.56$ (f) $176 \div 133$
(g) $9.01 \div 27.3$ (h) $4.6 \div 83.5$

4. (a) $\dfrac{77.2 \times 11.3}{12.1}$ (b) $\dfrac{6.8 \times 7.25}{4.6}$
(c) $\dfrac{2.8 \times 9.65}{3.7 \times 12.2}$ (d) $\dfrac{0.76 \times 4.2}{0.083 \times 161}$
(e) $\dfrac{83.5 \times 124 \times 0.51}{6.3 \times 96.4}$ (f) $\dfrac{22.2 \times 871 \times 76.2}{4.3 \times 974}$

5. (a) $(26.3)^2$ (b) $(9.7)^2$ (c) $(11.5)^2$ (d) $(127)^2$
(e) $(0.52)^2$ (f) $(0.073)^2$ (g) $(4.65)^2$ (h) $(7.86)^2$

6. (a) $\sqrt{17.3}$ (b) $\sqrt{9.85}$ (c) $\sqrt{124}$
(d) $\sqrt{0.56}$ (e) $\sqrt{763}$ (f) $\sqrt{0.082}$
(g) $\sqrt{0.0017}$ (h) $\sqrt{0.000\,565}$

EXERCISE 2 *Sets*

1. List the following sets:
(a) the even natural numbers less than 10
(b) the prime numbers between 10 and 30
(c) the multiples of 3 between 2 and 50
(d) the perfect squares between 2 and 101
(e) the perfect numbers between 1 and 10

A perfect number is an integer which is equal to the sum of all its factors except itself.
Example : $28 = 1 + 2 + 4 + 7 + 14$

2. State the solution set of the following.
(a) $\{3, 4, 5, 6\} \cap \{5, 6, 7\}$
(b) $\{a, b, c\}$ and $\{c, d, e\}$
(c) $\{3, 7, 9\} \cup \{9, 10, 12\}$
(d) $\{3, 4, 5\}$ or $\{6, 7, 8\}$
(e) $\{3, 4, 5\}$ and $\{6, 7, 8\}$
(f) $\{a, b, c\} \cap \{d, f, a\}$

EXERCISE 3 *Equations and inequations*

1. Solve and check.

(a) $3m - 4 = 6m + 5$

(b) $3(x - 2) = 2(x + 1)$

(c) $6(x - 1) - 2 = 5(x + 3) - 8$

(d) $6(t - 3) = 5(t + 2) - 8$

(e) $\dfrac{y - 2}{3} = \dfrac{3}{2}$

(f) $\dfrac{x + 1}{3} = 5$

(g) $\dfrac{2x - 3}{2} + \dfrac{x}{4} = \dfrac{3}{8}$

(h) $\dfrac{a + 2}{3} + \dfrac{a - 3}{2} = \dfrac{a + 5}{4}$

2. Solve and graph the solution set.

(a) $2x - 7 \geqq 3$

(b) $5x - 4 < 7x + 6$

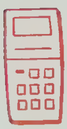
(c) $0 > 2x - 4$

(d) $2(2x + 1) - 3 \leqq 7x$

(e) $\dfrac{m + 3}{6} > \dfrac{m - 2}{3}$

(f) $\dfrac{4x - 3}{5} \leqq 3x + 6$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Find the distance between the following pairs of points.

1. (44.23, 66.89) and (21.26, 43.57)
2. (163.4, 287.6) and (142.6, 283.7)
3. (−5.713, 4.216) and (1.434, −2.745)
4. (16.3, −15.4) and (17.4, −11.8)
5. (0.713, 0.253) and (0.517, 0.173)
6. (−48.21, −66.81) and (−22.43, −81.77)
7. (0.0813, −0.0153) and (0.0716, −0.0216)
8. (444.2, 683.7) and (581.6, 763.2)
9. (−8.1, −9.3) and (−6.6, 7.8)
10. (78.73, −66.66) and (−84.32, −11.75)

Linear Systems

Graphical Solution of a Linear System

5.1 GRAPHING LINEAR EQUATIONS

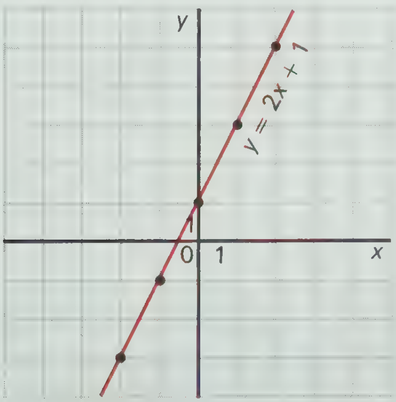
In Chapter 3 we graphed linear equations in the form $y = mx + b$ by first obtaining a table of values.

EXAMPLE 1. *Graph the relation $y = 2x + 1$, $x, y \in R$.*

Solution Obtain a table of values.

x	y
-2	-3
-1	-1
1	3

Plot the ordered pairs and draw the graph.



If the equation is not in the form $y = mx + b$, it is usually convenient to express it in this form before determining the ordered pairs.

EXAMPLE 2. *Graph the relation* $\{(x, y) \in R \times R \mid 3x - 2y = 1\}$.

Solution Express the relation in the form $y = mx + b$

$$3x - 2y = 1$$

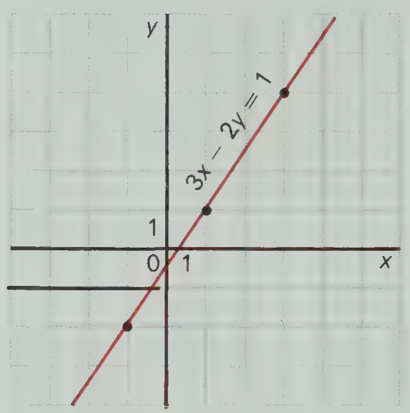
$$-2y = -3x + 1 \quad \text{subtracting } 3x \text{ from both sides}$$

$$2y = 3x - 1 \quad \text{multiplying both sides by } (-1)$$

$$y = \frac{3x - 1}{2} \quad \text{dividing both sides by 2}$$

Obtain a table of values and draw the graph.

x	y
-1	-2
1	1
3	4



Addition:
HERES
MERRY
XMAS
READER

EXERCISE 5-1

A **1.** Express the following in the form $y = mx + b$.

(a) $2x + y = 7$ (b) $4x + y + 7 = 0$ (c) $y - 3x = -4$
(d) $3 - 2x + y = 0$ (e) $3x + 2y = 4$ (f) $4x + 3y + 5 = 0$
(g) $x - y = 6$ (h) $5x - 3y = 5$ (i) $3y - 2x = 0$
(j) $2x - 5y - 1 = 0$

B **2.** Sketch the graphs of the following, where $x, y \in R$.

(a) $y = x + 2$ (b) $y = 3x - 1$ (c) $y = 2 - 3x$
(d) $y = 2 - 2x$ (e) $y = -4x$ (f) $y = \frac{1}{2}x + 3$
(g) $y = -2x - 3$ (h) $y = -\frac{1}{2}x - 1$
(i) $y = \dfrac{x + 1}{2}$ (j) $y = \dfrac{3 - x}{3}$

3. Sketch the graphs of the following relations by first expressing the relations in the form $y = mx + b$.

(a) $2x + y = 5$ (b) $2x - y = 1$ (c) $3x + 2y = 5$
(d) $x + 3y = 2$ (e) $x - 2y = 6$ (f) $3x - 3y + 1 = 0$
(g) $3y - x = 1$ (h) $x - 2y - 3 = 0$

It is often convenient to sketch relations by the "intercept method".

EXAMPLE 3. *Sketch the graph of the relation* $\{(x, y) \in R \times R \mid 2x - 3y = 6\}$.

Solution When $y = 0$, $2x - 3(0) = 6$

$$2x = 6$$

$$x = 3$$

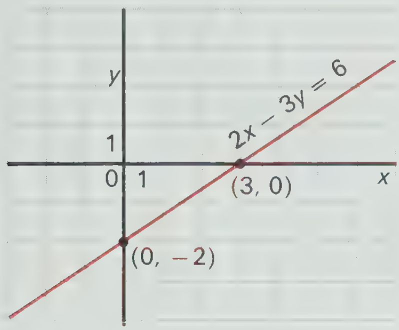
$\therefore$ (3, 0) is one ordered pair of the relation.

When $x = 0$, $2(0) - 3y = 6$

$$-3y = 6$$

$$y = -2$$

$\therefore$ (0, −2) is another ordered pair of the relation.



4. Use the *"intercept method"* to sketch the graphs of the following relations.

(a) $2x + y = 4$ (b) $x + 3y = 6$ (c) $3x + 2y = 12$
(d) $x - 2y = -4$ (e) $4x - y = 4$ (f) $2y - 3x = -6$
(g) $5x - 2y = 10$ (h) $3x - y - 6 = 0$ (i) $3x + 4y = 12$
(j) $5y - 3x + 15 = 0$

5.2 GRAPHICAL SOLUTION OF A LINEAR SYSTEM

We will now consider the graphical and algebraic solution of two linear equations. Such a pair of linear equations is called a **linear system of equations.**

INVESTIGATION 5.2

1. (a) Complete the table in your notes. y is determined by the equation $y = 2x - 1$.
(b) Graph the line defined by the equation $y = 2x - 1$.
(c) Each ordered pair in the table is called a solution of the equation. How many solutions does a linear equation have? Why?

2. Graph the line $y = -x + 5$, using the same pair of axes that you used in question 1.

3. A pair of linear equations such as $y = 2x - 1$ (from question 1) and $y = -x + 5$ (from question 2) is called a linear system of equations or, more briefly a linear system.

$y = 2x - 1$
(x, y)
(4, ▨)
(3, ▨)
(2, ▨)
(1, ▨)
(0, ▨)

(a) Use your graph from questions 1 and 2 to find a solution common to both equations of the system. (A solution of a linear system is a solution common to both equations of the system.)
(b) Check your solution in both equations of the system by substitution.

4. If the two equations determining a linear system define two distinct (different) lines, what is the maximum number of solutions for the system? Why?

5. If the equations determining a linear system define the same line, how many solutions does the system have?

6. Find a solution, if any exist, for the following linear systems by first graphing the lines determining the system. Be sure to check your solution in each equation of the system. Why check both?

(a) $y = 2x - 6$
$y = -\frac{1}{2}x + 4$

(b) $y = 2x + 1$
$y = -x - 5$

(c) $y = \frac{1}{3}x + 2$
$y = \frac{1}{3}x - 1$

7. If a linear system has no solution:
(a) What conclusions can be made about the lines from which the system was derived?
(b) How are the slopes of these lines related?

8. (a) By inspection, determine which of the following linear systems have no solution.

(i) $y = \frac{2}{3}x - 1$
$y = \frac{2}{3}x - \frac{1}{2}$

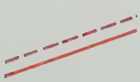
(ii) $y = 3x - 5$
$y = 2x - 5$

(iii) $2x + y = 5$
$6x + 3y = 7$

(b) Check your results in (a) by graphing.

consistent

inconsistent



dependent

Linear Systems
1. A linear system may have exactly one solution. This is true when the lines determining the system intersect at a point. Such a system is said to be **consistent.**
2. A linear system may have no solutions. This is true when the lines determining the system are parallel. Such a system is said to be **inconsistent.**
3. A linear system may have an unlimited number of solutions. This is true when the two equations defining the system determine the same straight line. Such a system is said to be **dependent.**

EXERCISE 5-2

1. Solve the following linear systems graphically. Check each solution by substitution.

(a) $y = \frac{1}{3}x + 3$
$y = -\frac{4}{3}x - 2$

(b) $y = \frac{3}{4}x - 1$
$y = x - 2$

(c) $y = -x + 4$
$y = -3x + 7$

(d) $y = -\frac{2}{3}x + 2$
$y = \frac{1}{2}x + 4$

Why are some graphical solutions inaccurate?

2. The FB Manufacturing Co. and the Toyto Novelty Co. both produce googles, the former according to the cost relation $C = 500 + 150n$ and the latter according to the cost relation $C = 800 + 100n$, where C is in

cents and n is the number of googles produced. By plotting C along the vertical axis and n along the horizontal axis, $n \geqq 0$, determine the following.
(a) On what size order would the FB Co. have the edge in competition?
(b) On what size order would the companies be even in competition? What is the cost for this size order?
(c) On what size order would the Toyto Co. have the edge in competition?

Insert signs to make 2 ▨ 3 ▨ 4 ▨ 5 = 5 a true statement.

Algebraic Solution of a Linear System

5.3 ELIMINATION BY COMPARISON

The graphical method of solving a linear system becomes difficult when the coordinates of the ordered pair of the solution set are not integers and cannot be easily read from the graph. Algebraic methods of solution are needed. The method of **comparison** will be studied in this section.

$$\boxed{\begin{aligned} \text{If } a &= b \\ \text{and } a &= c \\ \text{then } b &= c \end{aligned}}$$

EXAMPLE 1. *Solve* $y = 2x + 3$ ①
$y = x + 7$ ②

Solution Since $y = y$ for the point of intersection of the lines

$$\begin{aligned} \text{then } 2x + 3 &= x + 7 \\ \text{and } 2x - x &= 7 - 3 \\ x &= 4 \end{aligned}$$

$2x + 3 = y = x + 7$

Substitute in ① $y = 2x + 3$
$= 2(4) + 3$
$= 8 + 3$
$= 11$

Check in ② L.S. $= y$
$= 11$

R.S. $= x + 7$
$= 4 + 7$
$= 11$

$\therefore (x, y) = (4, 11)$

EXAMPLE 2. *Solve* $3x - 2y = 1$ ①
$2x - 3y = -1$ ②

Solution Express each equation in the form $y = mx + b$

① $\qquad\qquad\qquad\qquad$ ②

$$3x - 2y = 1 \qquad\qquad 2x - 3y = -1$$

$$-2y = -3x + 1 \qquad\qquad -3y = -2x - 1$$

$$y = \frac{3x}{2} - \frac{1}{2} \qquad\qquad y = \frac{2}{3}x + \frac{1}{3}$$

$$\text{or} \quad y = \frac{3x - 1}{2} \qquad\qquad \text{or} \quad y = \frac{2x + 1}{3}$$

Since $y = y$

$$\text{then } \frac{3x - 1}{2} = \frac{2x + 1}{3}$$

$$6 \times \left[\frac{3x - 1}{2}\right] = 6 \times \left[\frac{2x + 1}{3}\right] \qquad \text{L.C.D. for 2 and 3 is 6}$$

$$3(3x - 1) = 2(2x + 1)$$

$$9x - 3 = 4x + 2$$

$$5x = 5$$

$$x = 1$$

Substitute in ①

$$3x - 2y = 1$$

$$3(1) - 2y = 1$$

$$-2y = -2$$

$$y = 1$$

Check in ① and ②

L.S. $= 3x - 2y$ $\quad$ R.S. $= 1$

$= 3(1) - 2(1)$

$= 3 - 2$

$= 1$

L.S. $= 2x - 3y$ $\quad$ R.S. $= -1$

$= 2(1) - 3(1)$

$= 2 - 3$

$= -1$

$$(x, y) = (1, 1)$$

EXERCISE 5-3

A

1. Express the following in the form $y = mx + b$.

(a) $2x + y = 7$ $\quad$ (b) $-2x + y = 4$ $\quad$ (c) $3x + y + 7 = 0$

(d) $2x + 3y = 6$ $\quad$ (e) $4x - 2y = 7$ $\quad$ (f) $x + 3y - 7 = 0$

(g) $3y - 5x + 4 = 0$ $\quad$ (h) $3y + 2x = -4$ $\quad$ (i) $-2y - 3x = 7$

B

Solve and check the following.

2. (a) $y = 3x - 1$, $y = x + 5$ $\qquad$ (b) $y = 4x - 5$, $y = x + 7$

(c) $y = 2x + 1$, $y = x + 3$ $\qquad$ (d) $y = -2x + 3$, $y = x + 6$

3. (a) $y = -4x + 3$
$y = -2x - 7$

(b) $y = 4x + 3$
$y = -x - 7$

(c) $y = 2x + 3$
$y = 6 - 2x$

(d) $y = x + 3$
$y = 8 - 9x$

(e) $y = 2x + 5$
$y = 3$

(f) $y = -2x + 6$
$y = 0$

4. (a) $y = x + 2$
$y = \frac{1}{2}x + 3$

(b) $y = \frac{1}{2}x - 2$
$y = \frac{1}{4}x + 1$

(c) $y = \dfrac{x - 1}{2}$
$y = \dfrac{x + 2}{3}$

(d) $y = \dfrac{5x + 2}{3}$
$y = 2x + 1$

(e) $y = \dfrac{1 - 3x}{4}$
$y = \dfrac{2x - 3}{2}$

(f) $y = \frac{1}{2}x + 3$
$y = \dfrac{2x + 1}{3}$

5. (a) $2x + 3y = 5$
$4x + y = 5$

(b) $3x + 2y = 12$
$2x + 3y = 13$

(c) $3x + 4y = 2$
$4x - y = 9$

(d) $x + 3y = -5$
$3x - 2y = 7$

(e) $2x + 5y = 4$
$6x - 2y = -5$

(f) $2x + 3y = -2$
$2x - 3y = 0$

(g) $3x - 4y = -4$
$2y - 6x = 5$

(h) $5x - 8y = 8$
$10x + 4y = 1$

C 6. $x - 2y = -5$, $3x - y = 5$, and $2x + y = 0$ are the equations of three sides of a triangle. Find the coordinates of the vertices by solving the equations in pairs. Draw the figure.

Solve for x and y, mentally:
$3216x + 2722y = 5938$
$2722x + 3216y = 5938$

5.4 SOLVING EQUATIONS IN TWO VARIABLES BY ELIMINATION

Another algebraic method of solving two simultaneous equations is based on the elimination of one of the variables by addition or subtraction.

EXAMPLE 1. *Solve* $x + y = 5$ ①
$x - y = 1$ ②

Solution Add the equations vertically to eliminate one of the variables.

$$\begin{aligned} x + y &= 5 \quad ① \\ x - y &= 1 \quad ② \\ \text{adding: } 2x &= 6 \\ x &= 3 \end{aligned}$$

Substitute $x = 3$ in ①

$$\begin{aligned} x + y &= 5 \\ 3 + y &= 5 \\ y &= 2 \end{aligned}$$

Check in ① and ②

L.S. $= x + y$ R.S. $= 5$ L.S. $= x - y$ R.S. $= 3$

$= 3 + 2$ $= 5 - 2$

$= 5$ $= 3$

$\therefore x = 3, y = 2$

EXAMPLE 2. *Solve* $2x + y = 7$ ①
$x + y = 5$ ②

Solution In this case, vertical addition will not eliminate one of the variables. However, vertical subtraction will.

$$2x + y = 7 \quad ①$$
$$\underline{x + y = 5} \quad ②$$

Subtracting $\quad x \quad = 2$
(add the opposite)

Substitute $x = 2$ in ①

$$2x + y = 7$$
$$2(2) + y = 7$$
$$4 + y = 7$$
$$y = 3$$

Check in ① and ②

L.S. $= 2x + y$ R.S. $= 7$ L.S. $= x + y$ R.S. $= 5$

$= 2(2) + 3$ $= 2 + 3$

$= 4 + 3$ $= 5$

$= 7$

$\therefore x = 2, y = 3$

Test your skill:
200% of 30

EXAMPLE 3. *Solve* $4x - 3y = 2$ ①
$3x + 5y = 16$ ②

Neither addition nor subtraction eliminates one of the variables. Elimination will not occur unless either the "x" terms or "y" terms are identical or opposites. This can be achieved by multiplication.

Solution I
(eliminating x)

$$4x - 3y = 2 \quad ①$$
$$\underline{3x + 5y = 16} \quad ②$$

$3 \times ①$ $12x - 9y = 6$

$4 \times ②$ $\underline{12x + 20y = 64}$

Subtracting $-29y = -58$

$y = 2$

Solution II
(eliminating y)

$$4x - 3y = 2$$
$$\underline{3x + 5y = 16}$$

$5 \times ①$ $20x - 15y = 10$

$3 \times ②$ $\underline{9x + 15y = 48}$

Adding $29x = 58$

$x = 2$

Substitute in ①

$$4x - 3y = 2$$
$$4x - 3(2) = 2$$
$$4x - 6 = 2$$
$$4x = 8$$
$$x = 2$$

Substitute in ①

$$4x - 3y = 2$$
$$4(2) - 3y = 2$$
$$8 - 3y = 2$$
$$-3y = -6$$
$$y = 2$$

Check in ①

L.S. $= 4x - 3y$
$= 4(2) - 3(2)$
$= 8 - 6$
$= 2$
R.S. $= 2$

and ②

L.S. $= 3x + 5y$
$= 3(2) + 5(2)$
$= 6 + 10$
$= 16$
R.S. $= 16$

$\therefore$ The solution is (2, 2)

EXERCISE 5-4

B Solve and check.

1. (a) $3x + 2y = 13$, $3x - 2y = 5$ (b) $2x + y = 8$, $x + y = 5$ (c) $5x - y = 8$, $3x + 2y = 10$
2. (a) $5x + y = 15$, $2x - 3y = 6$ (b) $2x + 3y = 7$, $x + 2y = 4$ (c) $5x - 6y = 31$, $6x - 3y = 33$
3. (a) $7x + 3y = 27$, $2x + 5y = 16$ (b) $4x + 3y = 15$, $6x - 5y = -6$ (c) $3y + 9x = 42$, $2y - 4x = -2$

Solve

4. (a) $3x = 6 - 2y$, $6y = 5x + 30$ (b) $3y + 4x = 5$, $12x + 6y = 13$
(c) $3(y + 7) - 4(x + 6) = 0$, $7(y + 5) = 2(x + 10)$
5. (a) $x + y = 11$, $\frac{1}{2}x + \frac{1}{3}y = 5$ (b) $3x - y = 23$, $\frac{x}{3} + \frac{y}{4} = 4$
(c) $0.2x + 0.5y = 10$, $3x + 4y = 108$
6. (a) $\frac{2}{3}x - \frac{3}{2}y = -4$, $\frac{3}{2}x - \frac{y}{8} = 4$ (b) $\frac{x}{3} - \frac{y+1}{4} = \frac{3x+5y}{4} = x - 2$

5.5 PROBLEM SOLVING—PREPARATION

Many practical problems can be solved by first representing the information as algebraic equations. These equations may be solved using the methods of the previous sections. However, before attempting these problems, you must become familiar with the technique of representing words as algebraic expressions.

EXERCISE 5-5

B **1.** Represent each of the following as an algebraic expression.

(a) x plus five
(b) four more than x
(c) y minus three
(d) six times y
(e) 3 multiplied by b
(f) The sum of four and m
(g) two added to x
(h) five less than q
(i) the difference between r and seven
(j) b divided by 4
(k) the product of x and eight
(l) twice d
(m) x plus three times x
(n) $2x$ divided by y
(o) four times b increased by six
(p) five times the length increased by three
(q) four times the width decreased by ten
(r) Nicole's age two years from now
(s) Melissa's age three years ago
(t) double the speed less forty
(u) six times the price less four
(v) half the radius plus two
(w) Sam's age plus twice his age
(x) Twice Susan's age two years from now
(y) 5% of the cost of your text book

2. Represent the following as algebraic equations in one unknown.

(a) Twice the length is thirty-six.
(b) Three times John's age is twenty-seven.
(c) A number decreased by two is eight.
(d) A number increased by three is twelve.
(e) The width divided by five is twenty.
(f) Twice the width increased by two is twelve.
(g) Five times the height decreased by four is sixteen.
(h) Mary's age three years ago was seven.
(i) Double John's weight and add four and the result is eighty-four.
(j) One half the speed less ten is sixty.

3. Represent the following as algebraic equations in two unknowns. Let x and y represent the unknown quantities.

(a) The sum of two numbers is twenty-one.
(b) The length and width total forty.
(c) The difference between two numbers is one.
(d) Twice the width plus three times the length is seventy.
(e) One half the base increased by three equals four times the height.
(f) When twice one number is decreased by two-thirds of another number the result is five.
(g) A given number differs from half a smaller number by twelve.
(h) John has 15 tickets, some red and the rest yellow.
(i) Hockey tickets cost $3.00 for students and $5.00 for all others. The total gate receipts were $400.

5.6 PROBLEMS INVOLVING TWO UNKNOWNS

Steps in problem solving:

(a) Represent the unknown quantities by variables.
(b) Set up two equations using the conditions supplied in the problem.
(c) Solve the equations.
(d) Express your answer in a statement.
(e) Make an arithmetic check.

(i) NUMBER PROBLEMS

EXAMPLE 1. *The sum of 2 numbers is 25. Twice the first plus 3 times the second is 70. Find the numbers.*

Solution Let x represent the first number and y the second. (a)

$$x + y = 25 \quad ①$$
$$2x + 3y = 70 \quad ② \qquad \text{(b)}$$

① × 2 $\quad 2x + 2y = 50$

② × 1 $\quad 2x + 3y = 70$

Subtracting $\quad -y = -20$

$$y = 20 \qquad \text{(c)}$$

Substitute $y = 20$ in ①

$$x + y = 25$$
$$x + 20 = 25$$
$$x = 5$$

∴ The two numbers are 5 and 20. (d)

Check: (i) $5 + 20 = 25$ (the sum is 25) (e)
(ii) $(2 \times 5) + (3 \times 20) = 70$

EXERCISE 5-6

B

1. The sum of two numbers is eight. Three times the first plus 4 times the second is 29. Find the numbers.

2. The sum of two numbers is 9. Ten times the first plus eight times the second is 76. Find the numbers.

3. The difference between two numbers is five. Twice the larger plus six times the smaller is 26. Find the numbers.

4. Three times one number plus twice a second number is 29. Four times the first less 3 times the second is 16. Find the numbers.

5. Four times one number increased by one half another number is 24. Three times the first less the second is 7. Find the numbers.

If $3! = 3 \times 2 \times 1 = 6$ and $4! = 4 \times 3 \times 2 \times 1 = 24$

Evaluate: (a) $\dfrac{7!}{5!}$

(b) $\dfrac{8! \times 2!}{6!}$

(c) $\dfrac{8!}{4! \times 4!}$

(ii) MONEY PROBLEMS

EXAMPLE 2. *Tony had \$1.85 in dimes and quarters. If there were 11 coins in all, how many dimes did he have?*

Solution Let x represent the number of quarters
Let y represent the number of dimes

x quarters = $25x$ ¢
y dimes = $10y$ ¢
\$1.85 = 185 ¢

$$25x + 10y = 185 \quad (1)$$
$$x + y = 11 \quad (2)$$

$$(1) \quad 25x + 10y = 185$$
$$10 \times (2) \quad 10x + 10y = 110$$

Subtracting $\quad 15x = 75$

$$x = 5$$

Substitute $\quad x = 5$ in (2)

$$x + y = 11$$
$$5 + y = 11$$
$$y = 6$$

$\therefore$ He had six dimes.

Check : (i) $\$0.25 \times 5 + \$0.10 \times 6 = \$1.85$
(ii) 5 quarters + 6 dimes = 11 coins.

B

6. Kim paid \$3.95 for some football cleats. If he used dimes and quarters and there were 20 coins in all, how many dimes did he have?

7. A grade one class collected \$3.55 in nickels and dimes for the Red Cross. There were 56 coins. How many of each were there?

8. Tony bought a new suit for \$185. He paid the bill using \$5 bills and \$10 bills. If there were 26 bills in all, how many \$5 bills did he use?

9. Curt paid \$1760 for a used sports car. He used \$100 bills and \$20 bills to pay for it. If there were 28 bills in all, how many of each did he have?

10. While in Las Vegas, Lynda won \$29.50 playing the slot machines. Her winnings were in half dollars and silver dollars. If there were 42 coins, how many silver dollars did she win?

11. A sum of \$7.40 is made up of quarters and dimes. If there are 10 more quarters than dimes, how many coins of each kind are there?

C

12. Mr. Cane mixes jelly beans at 40¢/kg with saltwater taffy at 80¢/kg resulting in a 100 kg candy mixture worth 48¢/kg. How many kilograms of each candy are in the mixture?

13. A storekeeper wishes to mix tea worth 75¢/kg with tea worth 95¢/kg to make 100 kg of mixture valued at 86¢/kg. How many kilograms of each should she use?

(iii) PERCENT PROBLEMS

EXAMPLE 3. *Mark invested \$1500, part at 5% interest and part at 6%.*

If the interest on his investments totalled \$85, how much did he invest at each rate?

Solution Let \$$x$ be the amount invested at 5%
Let \$$y$ be the amount invested at 6%

$$x + y = 1500 \quad (1)$$

$$\frac{5}{100}x + \frac{6}{100}y = 85 \quad (2)$$

Total investment is \$1500

Interest on \$$x$ at 5% is 5% of x, or $\frac{5}{100} \times x$

Interest on \$$y$ at 6% is 6% of y, or $\frac{6}{100} \times y$

$$x + y = 1500 \quad (1)$$

100 × (2) to clear fractions $5x + 6y = 8500 \quad (2)$

5 × (1) $5x + 5y = 7500 \quad (1)$

$$5x + 6y = 8500 \quad (2)$$

Subtracting $-y = -1000$

$$y = 1000$$

Substitute $y = 1000$ in (1)

$$x + y = 1500$$

$$x + 1000 = 1500$$

$$x = 500$$

∴ He invested \$500 at 5% and \$1000 at 6%.

Check: (i) $500 + 1000 = 1500$ — Total investment was \$1500

(ii) 5% of \$500 is \$25
6% of \$1000 is \$60
\$25 + \$60 = \$85 — Total interest on investments is \$85

B **14.** Mary invested \$2000, part at 4% interest and part at 7%. If the interest on her investments totalled \$92, how much did she invest at each rate?

15. The interest on Sue's two investments totalled \$367. She had invested a total of \$3500, part at 11% and part at 10%. How much did she invest at each rate?

16. By investing \$5000, part at 8% and part at 9%, Terry received \$420 in interest. How much did he invest at each rate?

17. Frank invested \$200 and received \$13.50 in interest. Part of the \$200 was invested at 7%, the remainder at 6%. How much did he invest at each rate?

(iv) SPEED, DISTANCE, TIME PROBLEMS

$$\textbf{Speed} = \frac{\textbf{Distance}}{\textbf{Time}} \qquad v = \frac{s}{t}$$

$$\textbf{Distance} = \textbf{Speed} \times \textbf{Time} \qquad s = v \cdot t$$

$$\textbf{Time} = \frac{\textbf{Distance}}{\textbf{Speed}} \qquad t = \frac{s}{v}$$

It is usual to represent
Speed by v
Distance by s
Time by t

EXAMPLE 4. *On a motor trip of 260* km. *Sam drove part of the way at 70* km/h *and the remainder at 50* km/h. *If the total trip took 4* h, *how far did he drive at each rate?*

Solution Let the distance travelled at 70 km/h be x
Let the distance travelled at 50 km/h be y
Set up a distance, speed, time table.

s	v	t
x	70 km/h	$\frac{x}{70}$
y	50 km/h	$\frac{y}{50}$

Total distance is 260 km

$$x + y = 260 \quad (1)$$

Total time is 4 h

$$\frac{x}{70} + \frac{y}{50} = 4 \quad (2)$$

$$x + y = 260 \quad (1)$$

350 × (2) to clear fractions $5x + 7y = 1400 \quad (2)$

5 × (1) $5x + 5y = 1300 \quad (1)$

$$5x + 7y = 1400 \quad (2)$$

Subtracting $-2y = -100$

$$y = 50$$

Substitute $y = 50$ in (1)

$$x + y = 260$$

$$x + 50 = 260$$

$$x = 210$$

∴ He drove 210 km at 70 km/h and 50 km at 50 km/h.

Check: (i) $210 + 50 = 260$ Total distance is 260 km.

(ii) $\frac{210}{70} + \frac{50}{50} = 4$ Total time is 4 h

Insert signs to make
12 ▨ 2 ▨ 6 ▨ 1 = 0 a true
statement.

B

18. On a trip of 500 km, Mary drove part of the way at 60 km/h and the remainder at 50 km/h. If the total trip took 9 h, how far did she drive at each rate?

19. Terry drove for 3.5 h, part of the time at 80 km/h and part at 40 km/h. He covered a distance of 260 km. How far did he drive at each rate?

20. A car covered a total distance of 156 km averaging a speed of 40 km/h on highways and 18 km/h on country roads. The trip took 5 h. How many hours and how many kilometres did the car travel on country roads?

C

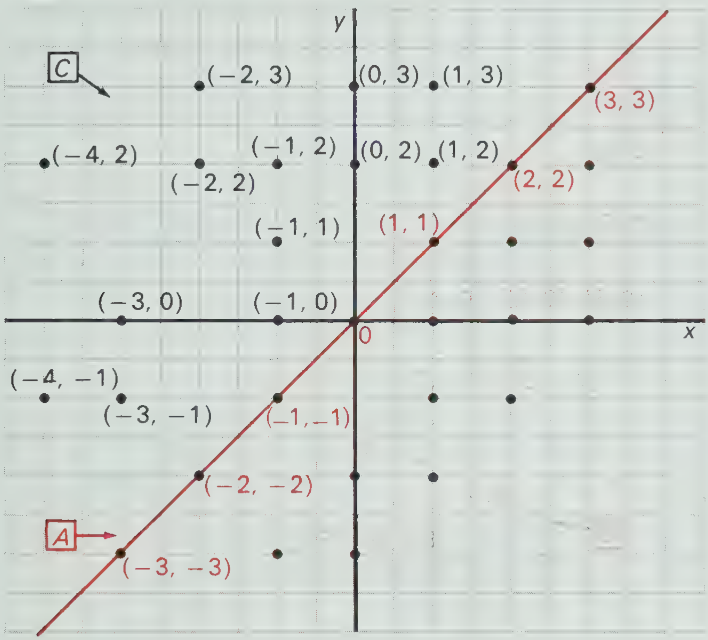
21. Two persons, 18 km apart, setting out at the same time meet in 4.5 h if they walk in the same direction, but in 2 h if they walk towards each

other. Find their rates of walking.

22. A commercial airliner travelled 2200 km with the wind in $7\frac{1}{3}$ h and returned the same distance against the wind in 10 h. Find (a) its air speed in still air (b) the wind velocity.

5.7 GRAPHING INEQUATIONS PART I

The graph of a relation divides the plane into three distinct regions.
Consider the relation $y = x$, $x, y \in R$.

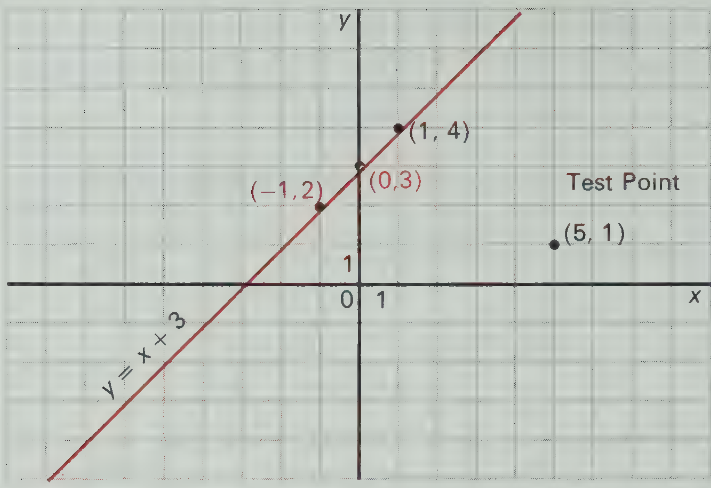


Region A: For every point here, $y = x$.
Region B: For every point here, $y < x$.
Region C: For every point here, $y > x$.

EXAMPLE 1. *Sketch the graph of $y = x + 3$ and indicate on the graph the regions*

(a) $y = x + 3$
(b) $y > x + 3$
(c) $y < x + 3$

Solution Obtain a few ordered pairs that satisfy the equation $y = x + 3$ and draw the graph.



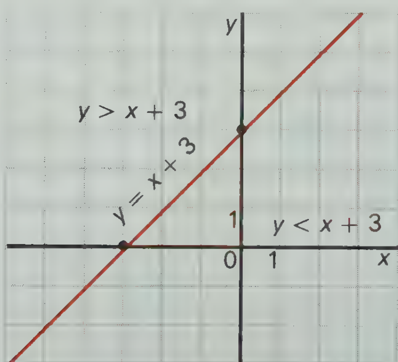
To decide where $y > x + 3$ and $y < x + 3$, take a test point not on the line.

In the area of the test point (5, 1),

when $y = 1$, L.S. $= 1$
when $x = 5$, R.S. $= 5 + 3$
$= 8$

$$\therefore \quad 1 < 8$$
$$\text{L.S.} < \text{R.S.}$$
$$y < x + 3$$

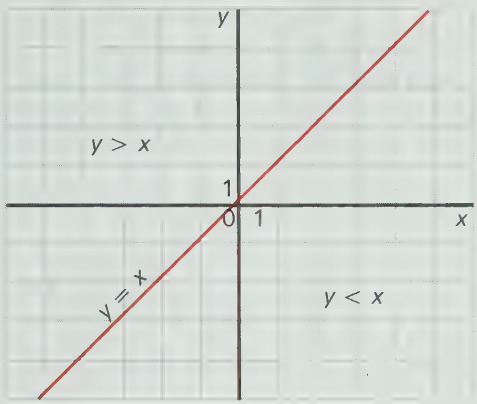
It is left to the student to take a test point on the other side of the line and show that in that region $y > x + 3$.



EXERCISE 5-7

For each of the following relations sketch a graph, clearly indicating the relations associated with each of the three regions formed.

1. $y = x + 1$
2. $y = x - 2$
3. $y = 2x + 1$
4. $y = 3x - 2$
5. $y = -x - 3$
6. $y = 4 - 3x$
7. $y = \dfrac{x + 1}{2}$
8. $2x + y = 3$
9. $2x - y = 4$
10. $3x - y = 5$

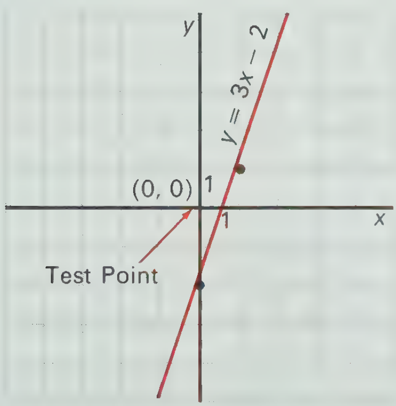


5.8 GRAPHING INEQUATIONS PART II

In the previous section we found that a line divides the plane into 3 regions. In this section you will be asked to graph 1 or 2 of these regions.

EXAMPLE 1. *Sketch the graph of* $y \leqq 3x - 2$.

Solution Draw the boundary line $y = 3x - 2$. Since equality shows the points on the line are included, draw a solid line. Use a test point to locate the required region.

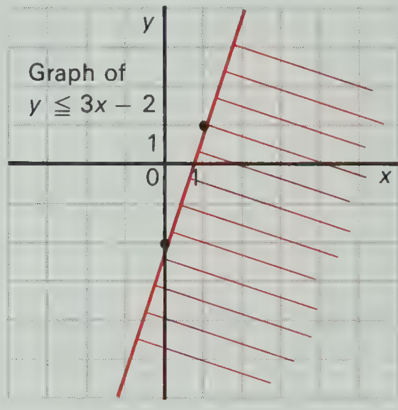


For the test point (0, 0)

$$\begin{aligned}
\text{L.S.} &= y \\
&= 0 \\
\text{R.S.} &= 3x - 2 \\
&= -2 \\
0 &> -2 \\
\text{L.S.} &> \text{R.S.}
\end{aligned}$$

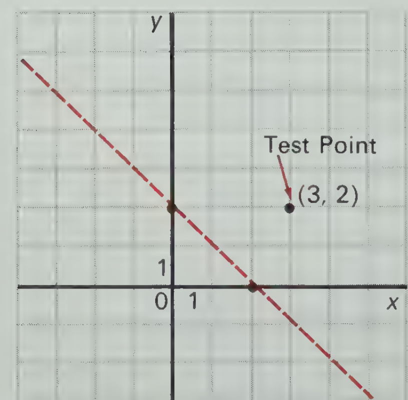
In the region of the test point, $y > 3x - 2$

$\therefore$ The region $y < 3x - 2$ lies on the other side of the line.



EXAMPLE 2. *Sketch the graph of $y > 2 - x$.*

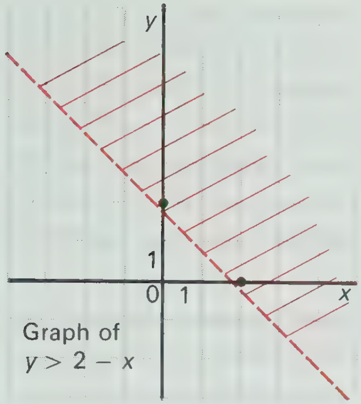
Solution Draw the boundary line $y = 2 - x$. Since there is no equality, the points on the line are not included. Draw a broken line.



For the test point (3, 2)

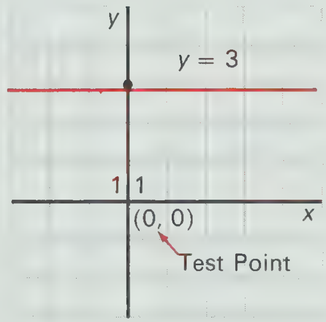
$$\begin{aligned}
\text{L.S.} &= y \\
&= 2 \\
\text{R.S.} &= 2 - x \\
&= -1 \\
2 &> -1 \\
\therefore \text{L.S.} &> \text{R.S.}
\end{aligned}$$

In the region of the test point, $y > 2 - x$.



EXAMPLE 3. *Sketch the graph of $y \geqq 3$.*

Solution Draw the boundary line $y = 3$.

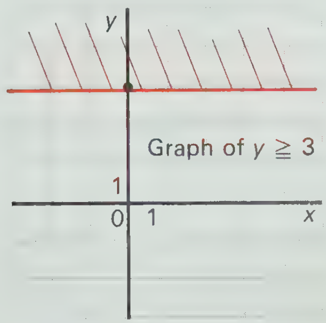


Why is the line solid?

For the test point (0, 0)

$$\begin{aligned} \text{L.S.} &= y \\ &= 0 \\ \text{R.S.} &= 3 \\ 0 &< 3 \\ \therefore \text{L.S.} &< \text{R.S.} \end{aligned}$$

In the region of the test point $y < 3$.



EXERCISE 5-8

Sketch the graph of each of the following for $(x, y) \in R \times R$.

B

1. (a) $y \geqq x + 3$ (b) $y < x - 2$
(c) $y > x + 1$ (d) $y \leqq x - 4$

2. (a) $y > 2x + 1$ (b) $y \leqq 3x - 1$
(c) $y \leqq 1 - x$ (d) $y > 2 - 2x$

3. (a) $y \leqq 3$ (b) $x \geqq 4$ (c) $x < -2$ (d) $y > -3$

4. (a) $y > 0$ (b) $x \geqq 0$ (c) $x \geqq -1$ (d) $y > -5$

First express the relations in the form
$y = mx + b$

5. (a) $y > \frac{1}{2}x + 1$ (b) $y \leqq \frac{x-1}{2}$ (c) $y \geqq 1 - \frac{1}{3}x$
(d) $y < \frac{2-x}{3}$ (e) $y > \frac{x+2}{2}$ (f) $y \geqq \frac{2}{3}x - 1$

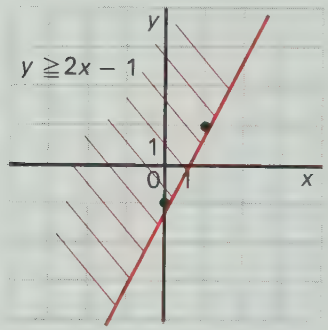
C 6. (a) $y - 2x < 3$ (b) $y - 3x > 1$
(c) $2y + x \geqq 3$ (d) $2x - y \leqq -1$
(e) $3y + 2x > 4$ (f) $2x - 3y \geqq 2$
(g) $x - 2y < 2$ (h) $2x + 3y - 1 > 0$

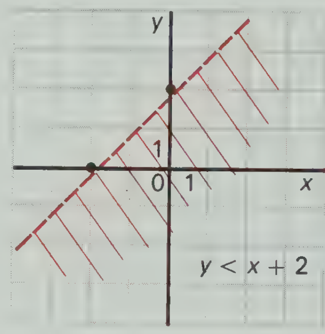
5.9 INTERSECTION AND UNION OF INEQUATIONS

In this section the set of points that satisfies two or more inequations will be dealt with.

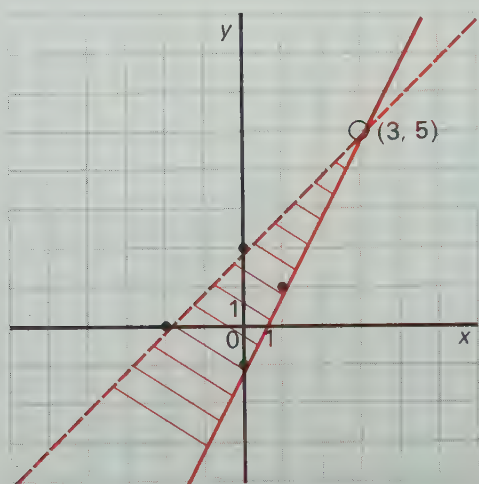
EXAMPLE 1. *Graph the solution set of* $\{(x, y) \in R \times R | y \geqq 2x - 1\} \cap \{(x, y) \in R \times R | y < x + 2\}$.

Solution Sketch the graph of each inequation.





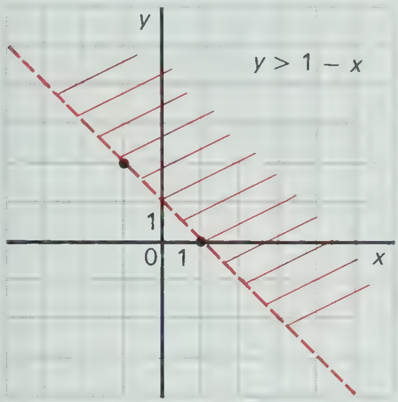
On the same set of axes, show the intersection of the two sets.

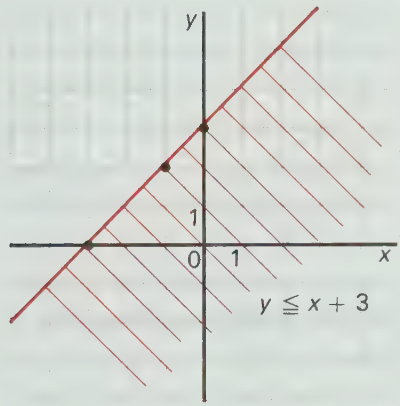


Why is the point (3, 5) not included in the solution set?

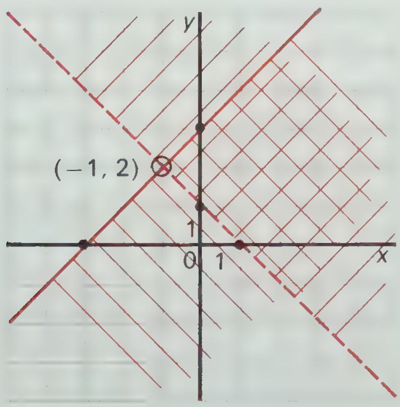
EXAMPLE 2. *Graph the solution set of* $\{(x, y) \in R \times R \mid y > 1 - x\} \cup \{(x, y) \in R \times R \mid y \leqq x + 3\}$.

Solution Sketch the graph of each inequation.





On the same set of axes, show the union of the two sets.

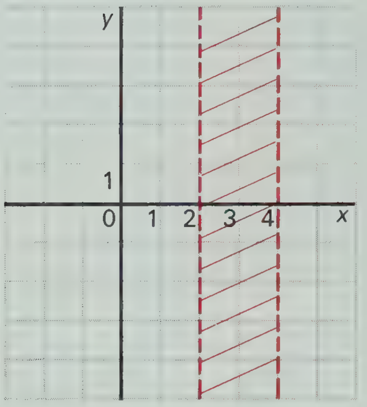


Why is the point $(-1, 2)$ included in the solution set?

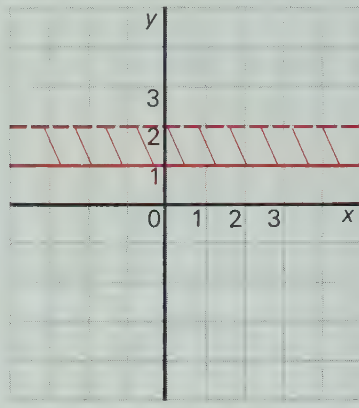
EXERCISE 5-9

A 1. State the defining relation for each of the following intersections.

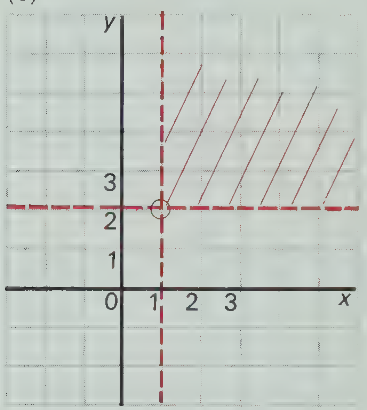
(a)



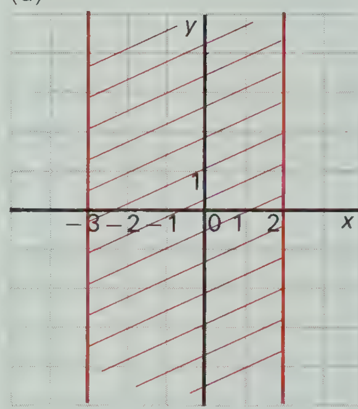
(b)



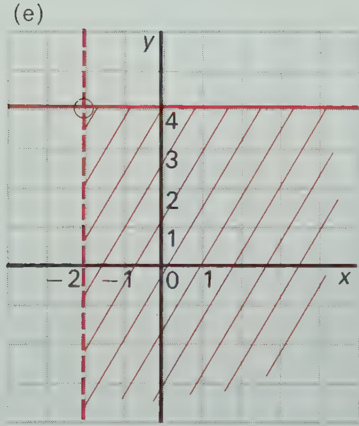
(c)



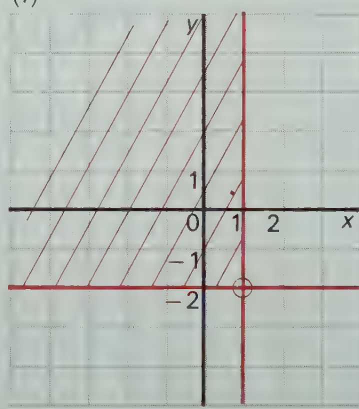
(d)



(e)



(f)



B Sketch the graphs of the following for $(x, y) \in R \times R$

2. (a) $\{y \geqq x + 1\} \cap \{y \leqq 3 - x\}$ (b) $\{y \leqq 2 - x\} \cap \{y \geqq 1 - 2x\}$
(c) $\{y > x - 3\} \cap \{y \leqq 1 - 3x\}$ (d) $\{y > 2x + 1\} \cap \{y < x - 2\}$
3. (a) $\{y < x + 1\} \cup \{y \geqq 1 - x\}$ (b) $\{y \geqq 3x + 1\} \cup \{y < 2x - 1\}$
(c) $\{y \geqq 1 + x\} \cup \{y \geqq 4 - 2x\}$ (d) $\{y < 1 - x\} \cup \{y < 2 + x\}$
4. (a) $\{x \geqq 3\} \cap \{y \leqq 1\}$ (b) $\{y > 1\} \cap \{x < -2\}$
(c) $\{y \leqq -2\} \cup \{x > 3\}$ (d) $\{y \leqq -2\} \cap \{x \geqq -3\}$
(e) $\{x > 0\} \cap \{y > -2\}$ (f) $\{y \leqq 0\} \cup \{x > -1\}$
5. (a) $\{y \geqq x + 1\} \cap \{y \leqq 3\}$ (b) $\{y < 1 - x\} \cap \{x \leqq 1\}$
(c) $\{y > 2x - 1\} \cup \{x \geqq 0\}$ (d) $\{y \leqq 2 - 3x\} \cap \{y > 0\}$
(e) $\{y \geqq \frac{1}{2}x - 1\} \cap \{x > 2\}$ (f) $\{y < 2x + 3\} \cup \{y < -1\}$
6. (a) $\{2x + y \geqq 0\} \cap \{y - x < 1\}$
(b) $\{y + 3x > 2\} \cup \{x + 2y \leqq 2\}$
(c) $\{3x + 2y < 6\} \cap \{x > -2\}$
(d) $\{\frac{1}{2}x + y \leqq 2\} \cap \{x \geqq 0\}$
(e) $\{x - y < 2\} \cap \{y \leqq 2\}$
(f) $\{2x - y \geqq -2\} \cup \{x + y \leqq 3\}$

Test your skill:
$11\frac{1}{4}\%$ of \$10 000.

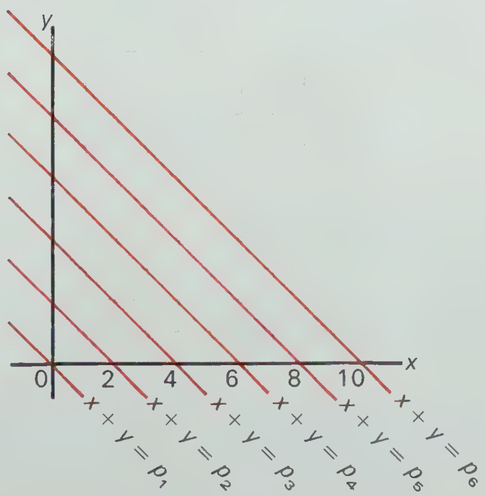
Express the relations in the form
$y = mx + b$

C 7. (a) $\{y \geqq 2x + 1\} \cap \{y \leqq 3\} \cap \{x \geqq -2\}$
(b) $\{y < x - 3\} \cap \{y \leqq 2 - x\} \cap \{y \geqq 5\}$
(c) $\{y < 2\} \cap \{x < 3\} \cap \{y > -1\} \cap \{x > -3\}$
(d) $\{2x + y < 2\} \cap \{y - x \leqq 3\} \cap \{y > -2\}$

5.10 LINEAR PROGRAMMING

Answer the following.

1. (a) By using the coordinates of a point (x, y) on each line in the diagram, calculate the values of p_1, p_2, p_3, p_4, p_5, and p_6.
(b) What property do all the lines graphed in the diagram have in common?
(c) What happens to the values of p as we choose lines further to the right?
(d) Which line yields the largest value for $x + y$?



When the graphical techniques of the previous section are applied to practical problems of business or industry, we say we are using techniques of linear programming. In linear programming problems it is required to find the maximum or minimum value of a linear expression such as $x + y$, where (x, y) must be a solution of a given linear system of inequations. The inequations of the system are often called constraints.

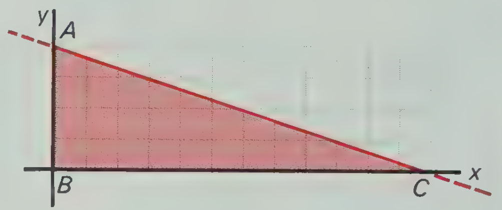
EXAMPLE 1. *Find the maximum value of the expression $x + y$, subject to the following conditions.*
(i) (x, y) must be on or below the line defined by $y = -\frac{1}{3}x + 4$.
(ii) (x, y) must be on or above the x-axis.
(iii) (x, y) must be on or to the right of the y-axis.

Solution We first graph the region defined by conditions (i), (ii), and (iii).

It can be shown that $x + y$ has a maximum at one of the vertices of $\triangle ABC$. We evaluate $x + y$ at each vertex.

We see that $x + y$ has a maximum value at vertex C when $x = 12$ and $y = 0$. The maximum value of $x + y$ is thus $12 + 0 = 12$.

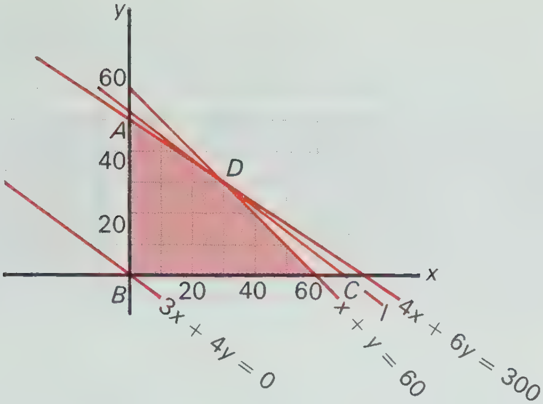
Vertex	Coordinates	$x + y$
A	(0, 4)	4
B	(0, 0)	0
C	(12, 0)	12



EXAMPLE 2. *A construction equipment company makes a light-duty four-wheel van that retails for \$3000 and a heavy-duty six-wheel van that retails for \$4000. Each day the plant receives 300 wheels and 60 engines that can be installed in either type of van. Find the number of units of each type that will result in the greatest possible sales volume.*

Solution Let x represent the number of light-duty vans and y the number of heavy-duty vans produced daily.

Production limits	Corresponding inequations
The number of vans of each type must be greater than or equal to zero.	$x \geqq 0$ $y \geqq 0$
The total number of wheels used must be less than or equal to 300.	$4x + 6y \leqq 300$
The total number of engines used must be less than or equal to 60.	$x + y \leqq 60$



Sales volume is $(3x + 4y)$ thousands of dollars. Draw the line $3x + 4y = 0$. Draw l, the line parallel to $3x + 4y = 0$ passing through a vertex of the region and farthest from the origin.

Note that the only point on l and belonging to the region is (30, 30), thus $3(30 + 4(30 = 90 + 120 = 210$. Thus the equation defining l is $3x + 4y = 210$. The maximum possible sales volume of \$210 000 occurs when 30 units of each type are made.

This result can be checked by substituting the coordinates of the vertices in the expression $3x + 4y$.

Vertex	Coordinates	Value of $3x + 4y$
A	(0, 50)	$0 + 200 = 200$
B	(0, 0)	0
C	(60, 0)	$180 + 0 = 180$
D	(30, 30)	$90 + 120 = 210$

Because (30, 30) yields the largest value for $3x + 4y$, our original conclusion is true.

EXERCISE 5-10

1. (a) Graph the region defined by the following linear system of inequations.

$$x \geqq 0,\ y \leqq -\tfrac{3}{2}x + 9$$
$$y \geqq 0,\ y \leqq -\tfrac{1}{2}x + 5$$

(b) Copy and complete the following table to determine the coordinates (x, y) of the solution to part (a) that makes the given expression a maximum.

Vertex	Solution (x, y)	Value of $x + 3y$	Value of $x + y$	Value of $3x + y$

2. A heating supply company manufactures two models of an air humidifier for which the market supply never meets the demand. Three different assembly machines are used in the manufacturing process according to the following schedule.

	Machine hours		
	Machine A	Machine B	Machine C
Model 1	1	2	$\frac{12}{7}$
Model 1A	2	1	$\frac{12}{7}$

If $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

evaluate (a) $\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix}$

(b) $\begin{vmatrix} -3 & 4 \\ -2 & 5 \end{vmatrix}$

(c) $\begin{vmatrix} \frac{1}{2} & \frac{2}{3} \\ \frac{3}{4} & \frac{5}{8} \end{vmatrix}$

Because of a shortage in skilled machine operators, machine operation is restricted to 12 h a day for each machine. The profit on model 1 is \$20 and on model 1A is \$30.
(a) Draw up a table as in Example 2, listing all production limits (constraints) together with their corresponding inequations.
(b) Graph the region bound by the production limits.
(c) How many units of each model should be made per day so that maximum profit is realized?

3. Two men, John Wallis and Bill Hoyle, make mathematical instruments for classroom use. In an hour John can make 10 protractors, 5 pairs of compasses, and 5 metre sticks. Bill can make 5 protractors, 5 pairs of compasses, and 20 metre sticks in the same time. The company they work for must fill an order for 40 protractors, 80 pairs of compasses, and 110 metre sticks. If John earns \$2.50/h and Bill \$3.00/h, find the number of hours each should work to fill the order at minimum labour cost.

4. A neighbourhood variety store keeps two brands of soap flakes in stock. The store has space for 240 boxes. It is known that brand X will have three times the sales volume of brand Y. If the profit per box for brand X is 10¢ and for brand Y is 12¢, how many boxes of each should the store stock for the maximum profit?

REVIEW EXERCISE

B **1.** Sketch the graphs of the following, where $(x, y) \in R \times R$.

(a) $y = 2x - 7$ (b) $y = -3x + 2$
(c) $2x - 5y = 10$ (d) $3y - 2x - 6 = 0$
(e) $3x = y - 10$ (f) $y = \dfrac{2x - 1}{3}$

2. Find the solution for each of the following linear systems.

(a) $2x + y = 3$, $5x - 2y = 3$ (b) $x + 2y = -3$, $3x - y = 5$ (c) $2x - y = -2$, $x - 2y = 2$
(d) $3x + 2y = 4$, $5x - 3y = 13$ (e) $3m - n = 3$, $3m - 5n = 3$ (f) $y = 2x - 7$, $y = 3 - 3x$
(g) $\frac{1}{5}x + \frac{1}{2}y = 5$, $x - y = 4$ (h) $\frac{a}{2} - \frac{b}{5} = 4$, $\frac{a}{7} + \frac{b}{15} = 3$
(i) $3(x - 1) + 2(y + 3) = 4$, $4(x + 1) - (y - 2) = 0$

3. The sum of two numbers is 18. Four times the first plus twice the second is 50. Find the numbers.

4. Three times a number plus twice a second number is 41. Four times the first plus five times the second is 71. Find the numbers.

5. Sam had \$1.90 in dimes and nickels. If there were 25 coins in all, how many dimes did he have?

6. Denise invested \$4000, part at 7% and part at 8%. The interest on her investment totalled \$310. How much did she invest at each rate?

7. The total interest on an investment was \$109. The investment was \$1100, part invested at 9% and the remainder at 11%. How much was invested at each rate?

8. On a trip of 355 km, Maria drove part of the way at 55 km/h and the remainder at 45 km/h. If the total trip took 7 h, how far did she drive at each rate?

9. Sketch the graph of each of the following for $(x, y) \in R \times R$.

(a) $y \geqq x - 2$ (b) $y < x + 5$ (c) $y > 2x - 1$
(d) $y \leqq 2 - x$ (e) $y < 3$ (f) $x > -1$
(g) $y \leqq 0$ (h) $y \geqq \dfrac{x - 1}{2}$ (i) $y < \frac{1}{2}x + 3$
(j) $x + y > 2$

10. Sketch the graphs of the following for $(x, y) \in R \times R$.

(a) $\{y \geqq x + 3\} \cap \{y \leqq 1 - x\}$ (b) $\{y > 2 + x\} \cup \{y < 4 - x\}$
(c) $\{y < 3 + x\} \cup \{y \geqq 2x + 1\}$ (d) $\{y < 3x + 2\} \cap \{y \geqq 2\}$
(e) $\{y \geqq 1 - 2x\} \cap \{x \geqq -1\}$ (f) $\{y < 1\} \cap \{x < 1\}$

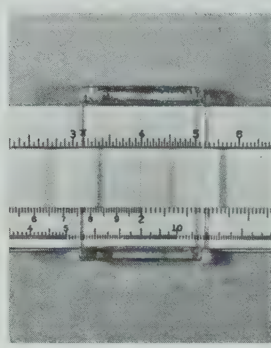
C **11.** Sketch the graphs of the following for $(x, y) \in R \times R$.

(a) $\{y \leqq 3x + 2\} \cap \{x \geqq -2\} \cap \{y \geqq -1\}$
(b) $\{x + 2y < 4\} \cap \{x > 0\} \cap \{y > 0\}$
(c) $\{x + y \leqq 3\} \cap \{y - x \leqq 3\} \cap \{x > -1\}$

REVIEW AND PREVIEW TO CHAPTER 6

SQUARE ROOT

Reading from A to D scales on the slide rule



Slide Rule

```
             5  6. 7  9
          ______________
          | 32|25.|75|00
       5  | 25
              ____
    10 6  |  725
             636
             ____
   112 7  |   8975
              7889
              ______
  1134 9  |   108600
              102141
              ______
                6459
```

A Formal Method

EXERCISE

1. Find the square roots to three figures reading from A to D scales:

(a) 169	(b) 2025	(c) 3364	(d) 8281	(e) 729
(f) 1225	(g) 3844	(h) 1764	(i) 361	(j) 576

2. Find the squares of the following to three figures reading from D to A scales:

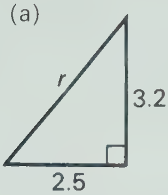
(a) 4.27	(b) 5.25	(c) 63.7	(d) 0.0125	(e) 3.08
(f) 85.5	(g) 3.14	(h) 6.34	(i) 25.5	(j) 16.7

3. Evaluate the following to three figures:

(a) 3.75^2	(b) $\sqrt{28.6}$	(c) $\sqrt{55.3}$	(d) 5.25^2
(e) 67.3^2	(f) 0.475^2	(g) $\sqrt{3.65}$	(h) $\sqrt{75\,500}$
(i) 0.259^2	(j) 5.37^2	(k) $\sqrt{66.6}$	(l) $\sqrt{32\,700}$
(m) 0.215^2	(n) 0.037^2	(o) $\sqrt{28.7}$	

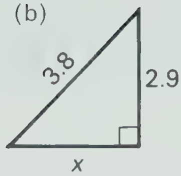
4. Evaluate as required:

(a)

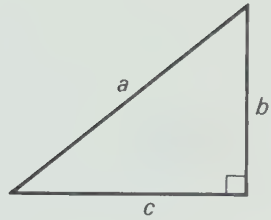


$k = \sqrt{2.5^2 + 3.2^2}$

(b)

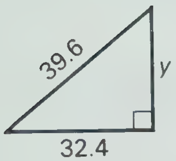


$x = \sqrt{3.8^2 - 2.9^2}$



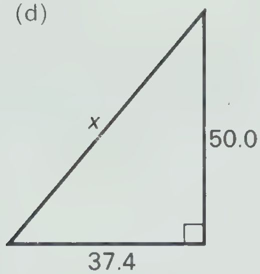
$a^2 = b^2 + c^2$

(c)



$x = \sqrt{50.0^2 + 37.4^2}$

(d)



$y = \sqrt{39.6^2 - 32.4^2}$

Evaluate the following.

1. $(6.374)^2$

2. $(0.8163)^3$

3. $(81.44)^4$

4. $(0.0816)^3$

5. $(6.813)^2 + (4.761)^3$

6. $(0.4167)^3 + (0.5843)^4$

7. $(1.683)^2 + (2.418)^3 - (8.713)^2$

8. $(2.481)^2(0.973)^3$

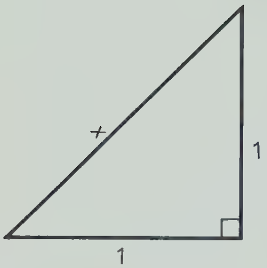
9. $(56.81)^4(2.78)^2$

10. $\dfrac{(1.813)^4}{(1.721)^3}$

CHAPTER 6

Radicals and Exponents

Numbers such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{8}$, are called radicals. In our previous work, when radicals appeared we found an approximation using tables, slide rule, or a formal method. In this chapter, we will add, subtract, multiply, and divide radicals before finding an approximation.



$x^2 = 1^2 + 1^2$
$= 1 + 1$
$= 2$

$x = \sqrt{2}$
$\doteq 1.414$
from tables.

6.1 MIXED AND ENTIRE RADICALS

Although 9 has two square roots, $+3$ and -3, when we write $\sqrt{9}$, we mean the positive (or principal) square root $+3$.

Since $\sqrt{25} = 5$ and $\sqrt{4} = 2$

then $\sqrt{25} \times \sqrt{4} = 5 \times 2$

$\sqrt{25 \times 4} = 10$

$\sqrt{100} = 10$

This result can be generalized to the law

$$\boxed{\sqrt{a} \times \sqrt{b} = \sqrt{ab}, \text{ where } a, b \geqq 0}$$

EXAMPLE 1. *Determine $\sqrt{5} \times \sqrt{3}$ to three significant digits.*

Method 1

$\sqrt{5} \times \sqrt{3} \doteq 2.236 \times 1.732$
$\doteq 3.873$

Method 2

$\sqrt{5} \times \sqrt{3} = \sqrt{15}$
$\doteq 3.873$

$\therefore \sqrt{5} \times \sqrt{3} \doteq 3.87$ to three significant digits.

Note that method 1 required multiplying 2.236 by 1.732, while method 2 required multiplying 5 by 3 which is much easier.

Radicals such as $\sqrt{18}$, $\sqrt{12}$, and $\sqrt{75}$ are called **entire radicals.** Radicals such as $3\sqrt{2}$, $2\sqrt{3}$, and $5\sqrt{3}$ are called **mixed radicals.**

EXAMPLE 2. *Express (a) $5\sqrt{2}$, (b) $4\sqrt{5}$ as entire radicals.*

Solution

(a) $5\sqrt{2} = \sqrt{25} \times \sqrt{2}$
$= \sqrt{25 \times 2}$
$= \sqrt{50}$

(b) $4\sqrt{5} = \sqrt{16} \times \sqrt{5}$
$= \sqrt{16 \times 5}$
$= \sqrt{80}$

EXAMPLE 3. *Express (a) $\sqrt{27}$, (b) $\sqrt{48}$ as mixed radicals in simplest form.*

Solution A radical is in *simplest* form when it has the smallest possible number under the radical sign.

(a) $\sqrt{27} = \sqrt{9} \times \sqrt{3}$

$= 3\sqrt{3}$

(b) $\sqrt{48} = \sqrt{16} \times \sqrt{3}$

$= 4\sqrt{3}$

$$\sqrt{48} = \begin{cases} \sqrt{48} \times \sqrt{1} \\ \sqrt{24} \times \sqrt{2} \\ \sqrt{16} \times \sqrt{3} \\ \sqrt{8} \times \sqrt{6} \\ \sqrt{12} \times \sqrt{4} \end{cases}$$

entire to mixed $\longrightarrow$

$\sqrt{8} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2}$

$\longleftarrow$ mixed to entire

EXAMPLE 4. *Simplify (a) $3\sqrt{2} \times 2\sqrt{7}$ (b) $2\sqrt{5} \times 3\sqrt{15}$*

Solution

Do these steps mentally

(a) $3\sqrt{2} \times 2\sqrt{7} = 3 \times 2 \times \sqrt{2} \times \sqrt{7}$

$= 6\sqrt{14}$

(b) $2\sqrt{5} \times 3\sqrt{15} = 6\sqrt{75}$

$= 6 \times \sqrt{25} \times \sqrt{3}$

$= 6 \times 5\sqrt{3}$

$= 30\sqrt{3}$

n	$\sqrt{n}$
1	1.0000
2	1.4142
21	[illegible]6
22	4.6904
23	4.7958
24	4.8990
25	5.0000
26	5.0990
27	5.1962
28	5.2915
29	5.3852
30	5.4772
31	5.5678
32	5.6569
33	5.7446
34	5.8310
35	5.9161
36	6.0000
37	6.0828
[illegible]	6.1644

EXERCISE 6-1

A 1. Evaluate

(a) $\sqrt{16}$ (b) $\sqrt{49}$ (c) $\sqrt{81}$ (d) $\sqrt{100}$ (e) $\sqrt{121}$

(f) $\sqrt{\frac{9}{16}}$ (g) $\sqrt{\frac{25}{36}}$ (h) $\sqrt{\frac{81}{100}}$ (i) $\sqrt{\frac{64}{81}}$ (j) $\sqrt{\frac{1}{25}}$

(k) $\sqrt{225}$ (l) $\sqrt{289}$ (m) $\sqrt{441}$ (n) $\sqrt{0.01}$ (o) $\sqrt{0.0025}$

2. Simplify

(a) $\sqrt{3} \times \sqrt{2}$ (b) $\sqrt{5} \times \sqrt{11}$ (c) $\sqrt{2} \times \sqrt{7}$

(d) $\sqrt{7} \times \sqrt{10}$ (e) $\sqrt{7} \times \sqrt{5}$ (f) $\sqrt{3} \times \sqrt{11}$

(g) $\sqrt{6} \times \sqrt{5}$ (h) $\sqrt{2} \times \sqrt{5}$ (i) $\sqrt{6} \times \sqrt{7}$

(j) $\sqrt{5} \times \sqrt{13}$ (k) $\sqrt{11} \times \sqrt{13}$ (l) $\sqrt{7} \times \sqrt{7}$

3. Simplify

(a) $2\sqrt{3} \times 3\sqrt{5}$ (b) $3\sqrt{7} \times \sqrt{2}$ (c) $4\sqrt{5} \times 2\sqrt{3}$

$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

$a\sqrt{b} \times c\sqrt{d}$

$= ac\sqrt{bd}$

(d) $\sqrt{3} \times 2\sqrt{7}$ (e) $3\sqrt{2} \times \sqrt{11}$ (f) $2\sqrt{5} \times 3\sqrt{3}$

(g) $2\sqrt{2} \times 3\sqrt{3}$ (h) $4\sqrt{7} \times 2\sqrt{5}$ (i) $4\sqrt{6} \times 3\sqrt{11}$

(j) $7\sqrt{2} \times 5\sqrt{3}$ (k) $6\sqrt{5} \times 2\sqrt{6}$ (l) $3\sqrt{2} \times 5\sqrt{2}$

B **4.** Change the following to mixed radicals in simplest form.

(a) $\sqrt{12}$ (b) $\sqrt{18}$ (c) $\sqrt{75}$ (d) $\sqrt{45}$

(e) $\sqrt{98}$ (f) $\sqrt{32}$ (g) $\sqrt{68}$ (h) $\sqrt{20}$

(i) $\sqrt{200}$ (j) $\sqrt{28}$ (k) $\sqrt{441}$ (l) $\sqrt{1024}$

(m) $\sqrt{72}$ (n) $\sqrt{50}$ (o) $\sqrt{8}$ (p) $\sqrt{24}$

5. Change the following to entire radicals.

(a) $2\sqrt{5}$ (b) $3\sqrt{7}$ (c) $3\sqrt{2}$ (d) $5\sqrt{2}$

(e) $3\sqrt{11}$ (f) $10\sqrt{5}$ (g) $10\sqrt{3}$ (h) $7\sqrt{2}$

(i) $5\sqrt{8}$ (j) $2\sqrt{14}$ (k) $7\sqrt{6}$ (l) $6\sqrt{11}$

(m) $20\sqrt{3}$ (n) $25\sqrt{2}$ (o) $4\sqrt{10}$ (p) $9\sqrt{2}$

6. Simplify

(a) $\sqrt{7} \times \sqrt{14}$ (b) $\sqrt{10} \times \sqrt{6}$ (c) $\sqrt{21} \times \sqrt{35}$

(d) $\sqrt{7} \times \sqrt{7}$ (e) $\sqrt{3} \times \sqrt{6}$ (f) $\sqrt{5} \times \sqrt{15}$

(g) $\sqrt{50} \times \sqrt{75}$ (h) $\sqrt{5} \times \sqrt{50}$ (i) $\sqrt{6} \times 3\sqrt{2}$

(j) $5\sqrt{7} \times 2\sqrt{14}$ (k) $3\sqrt{10} \times 2\sqrt{5}$ (l) $5\sqrt{3} \times 2\sqrt{15}$

(m) $\sqrt{6} \times \sqrt{3} \times \sqrt{2}$ (n) $\sqrt{5} \times \sqrt{2} \times \sqrt{15}$

(o) $\sqrt{10} \times \sqrt{15} \times \sqrt{6}$ (p) $3\sqrt{2} \times 2\sqrt{6} \times \sqrt{3}$

(q) $3\sqrt{5} \times 2\sqrt{3} \times 3\sqrt{5}$ (r) $2\sqrt{6} \times 3\sqrt{3} \times 4\sqrt{2}$

My age increased by 3 a gives a number which has an integral square root. Decrease my age by 3 a and you have the square root. How old am I?

EXAMPLE 5. *Simplify* $\sqrt{75x^3}$, $x \geqq 0$.

Solution $\sqrt{75x^3} = \sqrt{25x^2} \cdot \sqrt{3x}$

$= 5x\sqrt{3x}$

C **7.** Simplify ($x \geqq 0$)

(a) $\sqrt{25x^2}$ (b) $\sqrt{49x^4}$ (c) $\sqrt{x^4 \cdot x^2}$ (d) $3\sqrt{x^2}$

(e) $5x\sqrt{4x^2}$ (f) $\sqrt{3x} \cdot \sqrt{6x^3}$ (g) $\sqrt{5x^2} \cdot \sqrt{15x^3}$

(h) $\sqrt{6x} \cdot \sqrt{6x^3}$ (i) $\sqrt{45x^3}$ (j) $\sqrt{18x^4}$ (k) $\sqrt{20x}$

(l) $3x\sqrt{3x^2}$ (m) $5x\sqrt{5x^3}$ (n) $\sqrt{5x} \cdot \sqrt{3x}$ (o) $\sqrt{27x^3}$

(p) $\sqrt{8x^6}$ (q) $\sqrt{125x^4}$ (r) $5x\sqrt{5x^2}$ (s) $\sqrt{64x^3}$

(t) $\sqrt{99x^3}$

6.2 ADDITION AND SUBTRACTION OF RADICALS

Like terms such as $2a$ and $5a$ can be added using the distributive law.

$$2a + 5a = (2 + 5)a = 7a$$

Radicals such as $5\sqrt{2}$ and $3\sqrt{2}$ are called **like radicals** and can be added the same way.

$$5\sqrt{2} + 3\sqrt{2} = (5 + 3)\sqrt{2} = 8\sqrt{2}$$

EXAMPLE 1. *Simplify (a)* $5\sqrt{7} + 4\sqrt{7} - 3\sqrt{7}$

(b) $4\sqrt{5} + 7\sqrt{13} + 3\sqrt{5} - 2\sqrt{13}$

Solution

(a) $5\sqrt{7} + 4\sqrt{7} - 3\sqrt{7} = (5 + 4 - 3)\sqrt{7}$

$= 6\sqrt{7}$

(b) $4\sqrt{5} + 7\sqrt{13} + 3\sqrt{5} - 2\sqrt{13} = 4\sqrt{5} + 3\sqrt{5} + 7\sqrt{13} - 2\sqrt{13}$

$= 7\sqrt{5} + 5\sqrt{13}$

Note that $\sqrt{5}$ and $\sqrt{13}$ are not added because they are not like radicals.

EXAMPLE 2. *Simplify* $\sqrt{45} + \sqrt{5} - \sqrt{20}$.

Solution $\sqrt{45} + \sqrt{5} - \sqrt{20} = 3\sqrt{5} + \sqrt{5} - 2\sqrt{5}$

$= 2\sqrt{5}$

It is necessary to change all radicals to mixed radicals in simplest form before adding and subtracting.

EXERCISE 6-2

A

1. Simplify

(a) $3\sqrt{7} + 5\sqrt{7}$ (b) $6\sqrt{3} - 2\sqrt{3}$ (c) $8\sqrt{13} - 7\sqrt{13}$

(d) $3\sqrt{5} + 2\sqrt{5}$ (e) $2\sqrt{11} + 12\sqrt{11}$ (f) $12\sqrt{3} - 7\sqrt{3}$

(g) $3\sqrt{7} + \sqrt{7}$ (h) $8\sqrt{15} - 7\sqrt{15}$ (i) $7\sqrt{5} - \sqrt{5}$

(j) $8\sqrt{3} + 4\sqrt{3}$ (k) $\sqrt{5} + \sqrt{5}$ (l) $3\sqrt{3} - \sqrt{3}$

(m) $\sqrt{3} + 4\sqrt{3} + 7\sqrt{3}$ (n) $2\sqrt{7} - 5\sqrt{7} + 6\sqrt{7}$

2. Simplify

(a) $6\sqrt{3} - 2\sqrt{3} + 7\sqrt{11} - 9\sqrt{11}$

(b) $3\sqrt{17} + 8\sqrt{17} - 6\sqrt{15} + 3\sqrt{15}$

(c) $5\sqrt{2} + 3\sqrt{2} + 6\sqrt{3} - 2\sqrt{3}$

(d) $3\sqrt{5} + 7\sqrt{5} - 2\sqrt{7} + 5\sqrt{7}$

(e) $7\sqrt{3} + 2\sqrt{5} + 3\sqrt{3} + 4\sqrt{5}$

(f) $5\sqrt{7} - 3\sqrt{12} + 2\sqrt{7} - 5\sqrt{12}$

B **3.** Simplify

(a) $3\sqrt{2} + 5 + 2\sqrt{2} + 3$ (b) $6\sqrt{3} - 3 + 2\sqrt{3} + 7$

(c) $12 - 5\sqrt{2} - 3\sqrt{2} + 4$ (d) $3\sqrt{3} + 2\sqrt{5} + 2\sqrt{5} + \sqrt{3}$

(e) $5\sqrt{7} + 3 - 2\sqrt{7} - 3$ (f) $6\sqrt{2} - 2 - 3 - 7\sqrt{2}$

Insert brackets to make $2 \times 3 + 4 = 14$ a true statement.

4. Simplify

(a) $\sqrt{8} + \sqrt{18}$ (b) $\sqrt{12} + \sqrt{27}$ (c) $\sqrt{50} + \sqrt{18}$

(d) $\sqrt{98} - \sqrt{32}$ (e) $\sqrt{20} + \sqrt{45}$ (f) $\sqrt{75} - \sqrt{48}$

(g) $2\sqrt{18} + \sqrt{8} + 3\sqrt{2}$ (h) $5\sqrt{12} + 3\sqrt{12} - 3\sqrt{3}$

(i) $\sqrt{8} - 5\sqrt{2} + \sqrt{24}$ (j) $\sqrt{50} + 2\sqrt{18} - 10\sqrt{2}$

(k) $2\sqrt{48} + 5\sqrt{27}$ (l) $\sqrt{98} - 8\sqrt{2}$

(m) $\sqrt{50} + 2\sqrt{18}$ (n) $3\sqrt{8} - 2\sqrt{2}$

(o) $6\sqrt{32} + 2\sqrt{18} - 3\sqrt{50}$ (p) $3\sqrt{72} + \sqrt{75} - 3\sqrt{3}$

C **5.** Simplify ($x \geqq 0$, $a \geqq 0$)

(a) $3\sqrt{x} + 2\sqrt{x} - 4\sqrt{x}$ (b) $7\sqrt{a} + 2\sqrt{a} - 5\sqrt{a}$

(c) $3\sqrt{4x} + 5\sqrt{9x}$ (d) $2\sqrt{25a} - \sqrt{16a}$

(e) $2\sqrt{2x} + 4\sqrt{8x}$ (f) $3\sqrt{3a} - \sqrt{12a}$

6.3 MULTIPLICATION OF BINOMIAL RADICALS

Binomial radicals can be multiplied by applying the distributive property.

EXAMPLE 1. *Simplify (a)* $\sqrt{3}(\sqrt{5} - 2)$

(b) $2\sqrt{3}(\sqrt{6} + 3\sqrt{2})$

$a(b + c)$
$= ab + ac$

Solution

(a) $\sqrt{3}(\sqrt{5} - 2) = \sqrt{15} - 2\sqrt{3}$

(b) $2\sqrt{3}(\sqrt{6} + 3\sqrt{2}) = 2\sqrt{18} + 6\sqrt{6}$

$= 6\sqrt{2} + 6\sqrt{6}$

EXAMPLE 2. *Simplify* $(5\sqrt{2} - 4\sqrt{3})(3\sqrt{2} + 2\sqrt{3})$.

F O I L

$(5a - 4b)(3a + 2b)$
$= 15a^2 - 2ab - 8b^2$

Solution

$$\begin{aligned}(5\sqrt{2} - 4\sqrt{3})(3\sqrt{2} + 2\sqrt{3}) &= 15(2) + 10\sqrt{6} - 12\sqrt{6} - 8(3)\\ &= 30 - 2\sqrt{6} - 24\\ &= 6 - 2\sqrt{6}\end{aligned}$$

EXAMPLE 3. *Simplify* $(3\sqrt{5} - 2\sqrt{3})^2$

Solution

$$\begin{aligned}(3\sqrt{5} - 2\sqrt{3})^2 &= (3\sqrt{5} - 2\sqrt{3})(3\sqrt{5} - 2\sqrt{3})\\ &= 9(5) - 6\sqrt{15} - 6\sqrt{15} + 4(3)\\ &= 45 - 12\sqrt{15} + 12\\ &= 57 - 12\sqrt{15}\end{aligned}$$

EXAMPLE 4. *Simplify* $(2\sqrt{7} - \sqrt{5})(2\sqrt{7} + \sqrt{5})$.

$(2a - b)(2a + b)$
$= 4a^2 - b^2$

Solution

$$\begin{aligned}(2\sqrt{7} - \sqrt{5})(2\sqrt{7} + \sqrt{5}) &= 4(7) + 2\sqrt{35} - 2\sqrt{35} - (5)\\ &= 28 - 5\\ &= 23\end{aligned}$$

EXERCISE 6-3

B **1.** Simplify using the distributive property.

(a) $\sqrt{2}(\sqrt{3} + \sqrt{5})$
(b) $\sqrt{3}(5 + \sqrt{7})$
(c) $3(2\sqrt{5} - 3\sqrt{2})$
(d) $5(\sqrt{3} - 2\sqrt{5})$
(e) $\sqrt{3}(\sqrt{3} + 1)$
(f) $\sqrt{2}(2\sqrt{2} - \sqrt{6})$
(g) $\sqrt{2}(\sqrt{6} - 2\sqrt{3})$
(h) $3\sqrt{5}(\sqrt{2} + \sqrt{6})$
(i) $\sqrt{3}(2 + 3\sqrt{2})$
(j) $2\sqrt{3}(\sqrt{6} + \sqrt{18})$

2. Multiply the following binomial radicals.

(a) $(\sqrt{5} + 3)(\sqrt{5} + 4)$
(b) $(2\sqrt{3} + \sqrt{7})(\sqrt{3} + 3\sqrt{7})$
(c) $(\sqrt{3} + \sqrt{2})(2\sqrt{3} + \sqrt{2})$
(d) $(3\sqrt{2} - 2)(\sqrt{2} - 5)$
(e) $(2\sqrt{7} + 4\sqrt{3})(5\sqrt{7} + 6\sqrt{3})$
(f) $(8\sqrt{3} + 2\sqrt{5})(2\sqrt{3} - 7\sqrt{5})$
(g) $(3\sqrt{5} + \sqrt{3})(7\sqrt{5} + \sqrt{3})$
(h) $(5 + 3\sqrt{11})(7 - 6\sqrt{11})$
(i) $(4\sqrt{5} + 2\sqrt{2})(3\sqrt{5} - 6\sqrt{2})$
(j) $(5\sqrt{2} + 4\sqrt{3})(3\sqrt{7} - 2\sqrt{5})$

What happens to the area of a rectangle if you triple the length of the sides?

3. Square the following binomial radicals.

(a) $(\sqrt{5}+\sqrt{3})^2$ (b) $(\sqrt{7}-\sqrt{2})^2$ (c) $(\sqrt{6}-\sqrt{2})^2$

(d) $(\sqrt{7}-\sqrt{6})^2$ (e) $(3\sqrt{2}+1)^2$ (f) $(1-\sqrt{2})^2$

(g) $(4\sqrt{5}-\sqrt{2})^2$ (h) $(8\sqrt{3}+5\sqrt{6})^2$ (i) $(6\sqrt{10}+2\sqrt{3})^2$

(j) $2(\sqrt{3}+1)^2$ (k) $3(2\sqrt{2}+\sqrt{6})^2$ (l) $4(2\sqrt{5}-3)^2$

4. Multiply the following binomial radicals.

(a) $(\sqrt{5}+\sqrt{2})(\sqrt{5}-\sqrt{2})$ (b) $(\sqrt{2}+1)(\sqrt{2}-1)$

(c) $(3\sqrt{2}-4)(3\sqrt{2}+4)$ (d) $(2\sqrt{3}+2)(2\sqrt{3}-2)$

(e) $(\sqrt{7}+2\sqrt{3})(\sqrt{7}-2\sqrt{3})$ (f) $(4\sqrt{5}-\sqrt{2})(4\sqrt{5}+\sqrt{2})$

(g) $(5\sqrt{6}+2)(5\sqrt{6}-2)$ (h) $(5\sqrt{6}+\sqrt{2})(5\sqrt{6}-\sqrt{2})$

(i) $(8\sqrt{3}+7\sqrt{2})(8\sqrt{3}-7\sqrt{2})$ (j) $(3\sqrt{7}-4\sqrt{2})(3\sqrt{7}+4\sqrt{2})$

C 5. Simplify the following:

(a) $(\sqrt{3}+\sqrt{2})(2\sqrt{3}+5\sqrt{2})+(3\sqrt{2}-4\sqrt{3})^2$

(b) $(5+\sqrt{3})(2-4\sqrt{3})-(6\sqrt{3}+7)(2\sqrt{3}+5)$

(c) $(\sqrt{7}+\sqrt{3})(\sqrt{7}-\sqrt{3})-(\sqrt{11}+5)(\sqrt{11}-5)$

6. Simplify the following, $a \geqq 0$:

(a) $2\sqrt{a}(3\sqrt{a}-2)$ (b) $(\sqrt{a}+b)(\sqrt{a}-b)$

(c) $3\sqrt{a}(\sqrt{a}+a)$ (d) $(\sqrt{a}+2)^2$

(e) $\sqrt{2a}(\sqrt{2a}+\sqrt{a})$ (f) $(3\sqrt{a}+2)(2\sqrt{a}-5)$

6.4 DIVISION OF RADICALS

Division is the inverse operation of multiplication, so that if

$$\sqrt{3}\times\sqrt{5}=\sqrt{15} \quad \text{then} \quad \frac{\sqrt{15}}{\sqrt{5}}=\sqrt{3}$$

The result $\dfrac{\sqrt{15}}{\sqrt{5}}=\sqrt{3}=\sqrt{\dfrac{15}{3}}$ can be generalized as follows

$$\boxed{\frac{\sqrt{ab}}{\sqrt{b}}=\sqrt{\frac{ab}{b}}=\sqrt{a},\ b \neq 0}$$

also

$$\frac{\sqrt{ab}}{\sqrt{b}}=\frac{\sqrt{a}\sqrt{b}}{\sqrt{b}}=\sqrt{a}$$

EXAMPLE 1. *Simplify (a)* $\dfrac{\sqrt{10}}{\sqrt{5}}$ *(b)* $\dfrac{24\sqrt{21}}{6\sqrt{3}}$

Solution

(a) $\dfrac{\sqrt{10}}{\sqrt{5}} = \sqrt{\dfrac{10}{5}} = \sqrt{2}$ (b) $\dfrac{24\sqrt{21}}{6\sqrt{3}} = \dfrac{24}{6}\sqrt{\dfrac{21}{3}} = 4\sqrt{7}$

EXAMPLE 2. *Find $\dfrac{\sqrt{5}}{\sqrt{2}}$ correct to three figures.*

Solution

$$\frac{\sqrt{5}}{\sqrt{2}}$$

Method I

$$\frac{\sqrt{5}}{\sqrt{2}} \doteq \frac{2.236}{1.414} \doteq 1.581 \doteq 1.58$$

Method II

$$\frac{\sqrt{5}}{\sqrt{2}} = \sqrt{\frac{5}{2}} = \sqrt{2.5} \doteq 1.58$$

Method III

$$\frac{\sqrt{5}}{\sqrt{2}} = \frac{\sqrt{5}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{10}}{2} \doteq \frac{3.16}{2} \doteq 1.58$$

In method III, it was possible to change $\sqrt{2}$ to 2 by multiplying numerator and denominator by $\sqrt{2}$. This is called **rationalizing the denominator**.

EXAMPLE 3. *Find the value of $\dfrac{15\sqrt{7}}{2\sqrt{3}}$ correct to two decimal places by first rationalizing the denominator.*

Solution

$$\frac{15\sqrt{7}}{2\sqrt{3}} = \frac{15\sqrt{7}}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$
$$= \frac{15\sqrt{21}}{2(3)}$$
$$= \frac{5\sqrt{21}}{2}$$
$$\doteq \frac{5(4.583)}{2} = \frac{22.915}{2} \doteq 11.45$$

When we multiply two binomial radicals such as

$$(2\sqrt{7} - 3\sqrt{2})(2\sqrt{7} + 3\sqrt{2}) = 28 - 18 = 10$$

the result is a rational number. Such binomial radicals are called **conjugate radicals**.

$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$
$= a - b$

EXAMPLE 4. *Simplify* $\dfrac{4}{\sqrt{11}-\sqrt{3}}$ *by first rationalizing the denominator.*

Solution

$$\frac{4}{\sqrt{11}-\sqrt{3}} = \frac{4}{(\sqrt{11}-\sqrt{3})} \times \frac{(\sqrt{11}+\sqrt{3})}{(\sqrt{11}+\sqrt{3})}$$

$$= \frac{4(\sqrt{11}+\sqrt{3})}{11-3}$$

$$= \frac{4(\sqrt{11}+\sqrt{3})}{8}$$

$$= \frac{\sqrt{11}+\sqrt{3}}{2}$$

EXERCISE 6-4

A

1. Simplify

(a) $\dfrac{\sqrt{15}}{\sqrt{3}}$ (b) $\dfrac{\sqrt{75}}{\sqrt{3}}$ (c) $\dfrac{36\sqrt{18}}{9\sqrt{6}}$ (d) $\dfrac{\sqrt{24}}{\sqrt{6}}$

(e) $\dfrac{\sqrt{54}}{\sqrt{24}}$ (f) $\dfrac{2\sqrt{27}}{\sqrt{3}}$ (g) $\dfrac{\sqrt{50}}{\sqrt{2}}$ (h) $\dfrac{3\sqrt{98}}{\sqrt{2}}$

B

2. Use slide rule or square root tables to find the following to three-figure accuracy.

(a) $\dfrac{\sqrt{35}}{\sqrt{5}}$ (b) $\dfrac{\sqrt{18}}{\sqrt{3}}$ (c) $\dfrac{\sqrt{42}}{\sqrt{6}}$

(d) $\dfrac{15\sqrt{15}}{\sqrt{3}}$ (e) $\dfrac{20\sqrt{10}}{\sqrt{5}}$ (f) $\dfrac{8}{\sqrt{12}}$

(g) $\dfrac{\sqrt{21}}{\sqrt{3}}$ (h) $\dfrac{3\sqrt{50}}{\sqrt{2}}$ (i) $\dfrac{3\sqrt{75}}{\sqrt{15}}$

(j) $\dfrac{\sqrt{39}}{\sqrt{13}}$ (k) $\dfrac{3\sqrt{75}}{\sqrt{25}}$ (l) $\dfrac{3\sqrt{27}}{\sqrt{3}}$

(m) $\dfrac{\sqrt{18}+\sqrt{12}}{\sqrt{3}}$ (n) $\dfrac{15-\sqrt{75}}{5}$ (o) $\dfrac{9-\sqrt{45}}{3}$

3. Find the value of each of the following to three-figure accuracy by first rationalizing the denominator.

(a) $\frac{3}{\sqrt{5}}$ (b) $\frac{8}{\sqrt{7}}$ (c) $\frac{\sqrt{7}}{\sqrt{3}}$ (d) $\frac{2\sqrt{5}}{\sqrt{11}}$

(e) $\frac{\sqrt{5}}{3\sqrt{3}}$ (f) $\frac{3\sqrt{6}}{4\sqrt{2}}$ (g) $\frac{5\sqrt{8}}{4\sqrt{7}}$ (h) $\frac{3\sqrt{24}}{\sqrt{20}}$

(i) $\frac{\sqrt{3}}{\sqrt{45}}$ (j) $\frac{\sqrt{42}}{\sqrt{18}}$ (k) $\frac{5\sqrt{6}}{\sqrt{54}}$ (l) $\frac{3\sqrt{21}}{\sqrt{5}}$

(m) $\frac{7\sqrt{2}}{2\sqrt{11}}$ (n) $\frac{5\sqrt{3}}{4\sqrt{6}}$ (o) $\frac{7\sqrt{7}}{2\sqrt{2}}$

4. Simplify by first rationalizing the denominator.

(a) $\frac{1}{\sqrt{3}-\sqrt{2}}$ (b) $\frac{\sqrt{3}}{\sqrt{5}+\sqrt{2}}$ (c) $\frac{\sqrt{5}}{\sqrt{6}-\sqrt{2}}$

(d) $\frac{5}{\sqrt{2}-1}$ (e) $\frac{7}{2\sqrt{5}+\sqrt{2}}$ (f) $\frac{\sqrt{3}}{\sqrt{6}-1}$

(g) $\frac{5}{\sqrt{3}-4}$ (h) $\frac{5+\sqrt{2}}{5-\sqrt{2}}$ (i) $\frac{5+4\sqrt{3}}{5\sqrt{3}-3\sqrt{2}}$

(j) $\frac{3}{\sqrt{3}-\sqrt{2}}$ (k) $\frac{4}{\sqrt{7}-\sqrt{3}}$ (l) $\frac{5\sqrt{2}+\sqrt{3}}{5\sqrt{2}-\sqrt{3}}$

6.5 BASIC LAWS OF EXPONENTS

a^6 means $a \cdot a \cdot a \cdot a \cdot a \cdot a$

> a^n means $a \times a \times a \ldots$ to n factors
> a is the base , n is the exponent
> $a \in R$, $n \in N$

This definition helps us to state the following *three* basic laws of exponents.

Examples

$$a^2 \times a^3 = a \cdot a \times a \cdot a \cdot a = a^5$$

Basic Laws

$\boxed{x^m \cdot x^n = x^{m+n}}$

$$\frac{a^7}{a^3} = \frac{a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a}{a \cdot a \cdot a} = a \cdot a \cdot a \cdot a = a^4$$

$\boxed{x^m \div x^n = x^{m-n} \quad m > n}$

$$(a^2)^3 = a^2 \cdot a^2 \cdot a^2 = a^{2+2+2} = a^{3(2)} = a^6$$

$\boxed{(x^m)^n = x^{mn}}$

n	$\sqrt{n}$
1	1.0000
2	1.4142
3	1.7321
4	2.0000
5	2.2361
6	2.4495
7	2.6458
8	2.8284
9	3.0000
10	3.1623
11	3.3166
12	3.4641
13	3.6056
14	3.7417
15	3.8730
16	4.0000
17	4.1231
18	4.2426
19	4.3589
20	4.4721
21	4.5826
22	4.6904
23	4.7958
24	4.8990
25	5.0000
26	5.0990
27	5.1962
28	5.2915
29	5.3852
30	5.4772
31	5.5678
32	5.6569
33	5.7446
34	5.8310
35	5.9161
36	6.0000
37	6.0828
38	6.1644
39	6.2450
40	6.3246

EXAMPLE 1. *Simplify* (a) $5a^4 \times 6a^3$
(b) $15b^{12} \div 5b^7$
(c) $(5c^2)^3$

Solution

(a) $5a^4 \times 6a^3 = (5 \times 6) \times a^{4+3}$
$= 30a^7$

(b) $15b^{12} \div 5b^7 = \dfrac{15b^{12}}{5b^7}$
$= \dfrac{15}{5} \times b^{12-7}$
$= 3b^5$

(c) $(5c^2)^3 = 5^3(c^2)^3$
$= 125c^6$

The three basic laws can be used to develop two further properties.

Examples

$(ab)^3 = ab \cdot ab \cdot ab$
$= a \cdot a \cdot a \cdot b \cdot b \cdot b$
$= a^3b^3$

$(ab)^n = a^nb^n$

$\left(\dfrac{a}{b}\right)^4 = \dfrac{a}{b} \cdot \dfrac{a}{b} \cdot \dfrac{a}{b} \cdot \dfrac{a}{b}$
$= \dfrac{a^4}{b^4}$

$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

EXAMPLE 2. *Simplify* (a) $(2x^2)^3(2x^5)$ (b) $\dfrac{(3x^2)^3}{-3x}$

Solution

(a) $(2x^2)^3(2x^5) = 2^3 \times (x^2)^3 \times (2x^5)$
$= 8x^6 \times 2x^5$
$= 16x^{11}$

(b) $\dfrac{(3x^2)^3}{-3x} = \dfrac{3^3(x^2)^3}{-3x}$
$= \dfrac{27x^6}{-3x}$
$= -9x^5$

EXERCISE 6-5

A

1. Evaluate
(a) 2^4 (b) 3^2 (c) 2^5 (d) 3^3 (e) 4^3
(f) 5^3 (g) 9^2 (h) 7^2 (i) 10^3 (j) 3^4

2. Simplify
(a) $a^4 \times a^3$ (b) $a^2 \times a^3 \times a^4$ (c) $a \times a^5$
(d) $a^5 \times a^3$ (e) $b^5 \times b^6$ (f) $b^3 \times b^4 \times b^7$
(g) $b^3 \times b^7$ (h) $10^2 \times 10^3$

3. Simplify
(a) $(3a^2)(a^4)$ (b) $(5m^2)(2m^5)$ (c) $(4a^3)(2a^2)$
(d) $(4x^2)(x)$ (e) $(5n^2)(2n^3)$ (f) $(7x^4)(5x^3)$
(g) $(3x)(x^3)$ (h) $(2x^2)(3x^3)$ (i) $(3x^2)(2x^3)$

4. Simplify
(a) $6a^4 \div 3a$ (b) $9a^5 \div 3a^2$ (c) $t^7 \div t^6$
(d) $9x^3 \div 3x$ (e) $12m^3 \div 4m^2$ (f) $6n^3 \div 3n^2$

(g) $32a^5 \div 8a^4$ (h) $9t^6 \div 9t^4$ (i) $21x^5 \div 7x$

5. Simplify

(a) $(2^3)^2$ (b) $(3^2)^2$ (c) $(m^5)^3$ (d) $5(n^2)^7$

(e) $(2p^3)^4$ (f) $(5a)^4$ (g) $(3x^3)^3$ (h) $(2x^2)^3$

(i) $(-1)^5$ (j) $(-1)^{27}$ (k) $(-1)^{125}$ (l) $(-1)^{128}$

6. Simplify

(a) $\dfrac{(3a^2b^2)(7a^4b^3)}{21a^3b^3}$ (b) $\dfrac{(6a^2b^2)(4a^2b)}{12ab^2}$ (c) $\dfrac{(5a^4b^3)(4ab^2)}{10ab}$

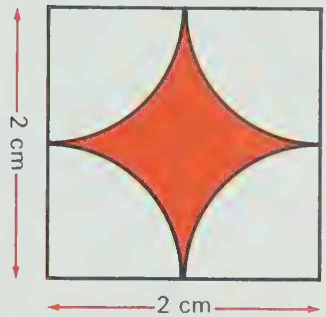
(d) $\dfrac{(7a^2b^2)(3a^4b^3)}{7a^4b^4}$ (e) $\dfrac{(3ab)(5ab^2)}{15a^2b^2}$ (f) $\dfrac{24m^3n^4}{(3m^2)(4n^2)}$

7. Simplify

(a) $2^a \times 2^b \times 2^c$ (b) $2^{a+b} \times 2^{a+2b}$ (c) $2^{a+b} \div 2^a$

(d) $\dfrac{3^a \times 3^b \times 3^c}{3^{a+b}}$ (e) $\dfrac{3^{abc}}{(3^a)^b}$ (f) $\dfrac{a^b \times a^{a+b}}{a^{a+1}}$

Find the area of the shaded region.



6.6 ZERO AND NEGATIVE EXPONENTS

In the previous section, we worked with powers where the exponents were natural numbers. We now use the basic laws to give meaning to powers with zero and negative exponents.

Question **Conclusion**

$$a^5 \div a^5 = \frac{a^5}{a^5} = \frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a \cdot a \cdot a \cdot a} = 1 \rightarrow a^0 = 1$$

$$a^5 \div a^5 = a^{5-5} = a^0 \rightarrow a^0 = 1$$

The above example (worked two ways) suggests the general rule:

$$\boxed{x^0 = 1,\ x \neq 0}$$

for any *non-zero* value of x.

Question **Conclusion**

$$a^5 \div a^8 = \frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a} = \frac{1}{a \cdot a \cdot a} = \frac{1}{a^3} \rightarrow a^{-3} = \frac{1}{a^3}$$

$$a^5 \div a^8 = a^{5-8} = a^{-3} \rightarrow a^{-3} = \frac{1}{a^3}$$

This example (worked two ways) suggests the general rule:

$$\boxed{x^{-n} = \frac{1}{x^n} \quad \text{or} \quad x^n = \frac{1}{x^{-n}}, \quad (x \neq 0)}$$

EXAMPLE 1. *Simplify (a)* 4^{-2} *(b)* $\dfrac{1}{3^{-4}}$

$a^3 = aaa$
$a^2 = aa$
$a^1 = a$
$a^0 = 1$
$a^{-1} = \frac{1}{a}$
$a^{-2} = \frac{1}{a^2}$
$a^{-3} = \frac{1}{a^3}$

Solution

(a) $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$ (b) $\frac{1}{3^{-4}} = 3^4 = 81$

EXAMPLE 2. *Simplify (a)* $\frac{2 \times 5^{-3}}{10^{-2}}$ *(b)* $(-2)^{-4}$

Solution

(a) $\frac{2 \times 5^{-3}}{10^{-2}} = \frac{2 \times 10^2}{5^3}$

$= \frac{2 \times 100}{125}$

$= \frac{2 \times 4}{5}$

$= \frac{8}{5}$

(b) $(-2)^{-4} = \frac{1}{(-2)^4}$

$= \frac{1}{16}$

EXAMPLE 3. *Simplify (a)* $9a^5 \div 3a^{-3}$ *(b)* $3^{-1} + 2^{-2}$

Solution

(a) $9a^5 \div 3a^{-3} = \frac{9}{3} \times a^{5-(-3)}$

$= 3a^{5+3}$

$= 3a^8$

(b) $3^{-1} + 2^{-2} = \frac{1}{3} + \frac{1}{2^2}$

$= \frac{1}{3} + \frac{1}{4}$

$= \frac{4+3}{12}$

$= \frac{7}{12}$

EXERCISE 6-6

A

1. Evaluate

(a) 5^{-2} (b) 3^0 (c) $(-3)^0$ (d) (3^{-4}) (e) 2^{-3}
(f) 3^{-2} (g) $5^0 \times 2^0$ (h) 10^{-3} (i) $(-5)^0$ (j) $(0.1)^{-1}$

2. Simplify

(a) $a^3 \cdot a^{-5}$ (b) $a^{-1} \cdot a^7$ (c) $a^0 \cdot a^4$ (d) $a^{-1} \cdot a^{-3}$
(e) $a^{10} \cdot a^4 \cdot a^{-5}$ (f) $x \cdot x^0 \cdot x^3$ (g) $5^{-3} \times 5^2 \times 5$ (h) $(2^3)^0$
(i) $\frac{b^5}{b^6}$ (j) $b^7 \div b^{10}$ (k) $b^{-3} \div b^2$ (l) $b^{-2} \div b^{-3}$

B

3. Simplify

(a) $3a^5 \times 5a^{-2}$ (b) $(x^2y^5)(x^3y^{-8})$ (c) $(2m^5)(3m^{-5})$
(d) $7a^{-2} \times 3a^3$ (e) $(3a^{-3})(2a^{-2})$ (f) $(5a^2)(3a^{-4})$
(g) $5a^2 \times 2a^{-2}$ (h) $(3ab)(2a^{-1}b^{-2})$ (i) $(6a^2b^2)(2a^{-3}b^3)$

4. Simplify

(a) $a^3 \div a^{-5}$ (b) $m^{-3} \div m^2$ (c) $r^7 \div r^{10}$
(d) $27a^2 \div 3a^{-5}$ (e) $4p^5 \div 12p^8$ (f) $7t^3 \div 14t^5$
(g) $(3x^{-5})^{-1}$ (h) $(x^{-3})^{-2}$ (i) $(2a^3)^{-3}$

5. Evaluate

(a) $2^0 + 2^{-1}$ (b) $2^{-1} + 3^{-1}$ (c) $(3^{-1})^2$
(d) $(2^{-3} \times 2^2)^2$ (e) $(3^{-1})^{-1}$ (f) $(\frac{1}{2})^{-1}$

(g) $(\frac{3}{4})^0$ (h) $4^{-1} + 2^{-2}$ (i) $3^{-1} + 1^{-3}$

(j) $(\frac{1}{10})^{-1}$ (k) $(0.01)^{-1}$ (l) $\dfrac{2^{-3} \times 5^2}{3^{-2}}$

6.7 RATIONAL EXPONENTS

We have defined $x^3 = x \cdot x \cdot x$, and followed with $x^0 = 1$, and $x^{-1} = \dfrac{1}{x}$, $(x \neq 0)$. What meaning can we give to

$$\boxed{x^{\frac{1}{2}} = ?}$$

Again, we can use the basic laws to find a meaning for rational exponents.

$$25^{\frac{1}{2}} \times 25^{\frac{1}{2}} = 25^{\frac{1}{2}+\frac{1}{2}} = 25^1 = 25$$

$$\sqrt{25} \times \sqrt{25} = 5 \times 5 = 25$$

$$\therefore 25^{\frac{1}{2}} = \sqrt{25}$$

$$x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = (x^{\frac{1}{2}})^2 = x^1$$

$25^{\frac{1}{2}}$
$= \sqrt{25}$
$= 5$

So that by taking the square root of both sides of $(x^{\frac{1}{2}})^2 = x$, we have

$$\sqrt{(x^{\frac{1}{2}})^2} = \sqrt{x^1}, \text{ or}$$

$$\boxed{x^{\frac{1}{2}} = \sqrt{x},\ x \geq 0}$$

In the same way,

$$x^{\frac{1}{5}} \cdot x^{\frac{1}{5}} \cdot x^{\frac{1}{5}} \cdot x^{\frac{1}{5}} \cdot x^{\frac{1}{5}} = x$$

$$(x^{\frac{1}{5}})^5 = x$$

Taking the fifth root of both sides,

$$\sqrt[5]{(x^{\frac{1}{5}})^5} = \sqrt[5]{x}, \text{ or}$$

$$\boxed{x^{\frac{1}{5}} = \sqrt[5]{x}}$$

We generalize this result to

$$\boxed{x^{1/r} = \sqrt[r]{x},\ x \geq 0,\ r \neq 0}$$

Further

$$\boxed{x^{p/r} = \begin{cases} (x^p)^{1/r} = \sqrt[r]{x^p} \\ (x^{1/r})^p = (\sqrt[r]{x})^p \end{cases}}$$

If $\sum_{n=1}^{5} n = 1 + 2 + 3 + 4 + 5$
$= 15$

find: (a) $\sum_{n=1}^{10} n$

(b) $\sum_{n=4}^{10} n$

(c) $\sum_{n=90}^{100} n$

EXAMPLE 1. *Simplify (a)* $64^{\frac{1}{3}}$ *(b)* $81^{-\frac{1}{4}}$ *(c)* $125^{\frac{2}{3}}$

Solution

(a) $64^{\frac{1}{3}} = \sqrt[3]{64}$

$= 4$

(b) $81^{-\frac{1}{4}} = \dfrac{1}{81^{\frac{1}{4}}}$

$= \dfrac{1}{\sqrt[4]{81}}$

$= \frac{1}{3}$

(c) $125^{\frac{2}{3}} = (\sqrt[3]{125})^2$

$= 5^2$

$= 25$

EXAMPLE 2. *Evaluate (a)* $16^{-\frac{3}{2}}$ *(b)* $27^{0.\dot{3}}$ *(c)* $10^{-0.5} \times 10^{1.5}$

Solution

(a) $16^{-\frac{3}{2}} = \dfrac{1}{16^{\frac{3}{2}}}$

$= \dfrac{1}{(\sqrt{16})^3}$

$= \dfrac{1}{4^3}$

$= \dfrac{1}{64}$

(b) $27^{0.\dot{3}} = 27^{\frac{1}{3}}$

$= \sqrt[3]{27}$

$= 3$

(c) $10^{-0.5} \times 10^{1.5} = 10^{-0.5+1.5}$

$= 10^{1.0}$

$= 10$

EXERCISE 6-7

A

1. Evaluate

(a) $9^{\frac{1}{2}}$ (b) $8^{\frac{1}{3}}$ (c) $64^{\frac{1}{3}}$ (d) $125^{\frac{1}{3}}$
(e) $81^{\frac{1}{2}}$ (f) $81^{\frac{1}{4}}$ (g) $625^{\frac{1}{2}}$ (h) $625^{\frac{1}{4}}$

B

2. Evaluate

(a) $81^{-\frac{1}{4}}$ (b) 5^{-2} (c) $16^{-\frac{1}{2}}$ (d) $16^{-\frac{1}{4}}$
(e) $4^{-\frac{1}{2}}$ (f) $36^{-\frac{1}{2}}$ (g) $27^{\frac{2}{3}}$ (h) $8^{\frac{2}{3}}$
(i) $9^{\frac{3}{2}}$ (j) $16^{\frac{3}{4}}$ (k) $25^{\frac{3}{2}}$ (l) $8^{-\frac{2}{3}}$

3. Evaluate

(a) $27^{\frac{1}{3}}$ (b) $64^{-\frac{5}{6}}$ (c) $4^{\frac{3}{2}}$ (d) $32^{-\frac{3}{5}}$
(e) $\sqrt[3]{64^2}$ (f) $125^{\frac{4}{3}}$ (g) $\sqrt[4]{16^3}$ (h) $(\frac{1}{4})^{-\frac{1}{2}}$
(i) $32^{-\frac{4}{5}}$ (j) $32^{\frac{2}{5}}$ (k) $27^{-\frac{4}{3}}$ (l) $(\frac{1}{81})^{-\frac{1}{4}}$

Simplify
$\dfrac{8^{-\frac{2}{3}} \times 8^{\frac{1}{2}}}{8^{\frac{1}{2}} \div 8^{-\frac{2}{3}}}$

4. Simplify

(a) $2^{\frac{3}{5}} \times 2^{\frac{2}{5}}$ (b) $10^{0.5} \times 10^{0.25}$ (c) $10^{0.5} \div 10^{0.25}$
(d) $10 \times 10^{0.\dot{3}} \div 10^{\frac{2}{3}}$ (e) $5^{\frac{4}{3}} \div 5^{\frac{1}{3}}$ (f) $2^{0.3} \times 2^{0.25} \times 2^{0.45}$
(g) $(81^2)^{\frac{1}{4}}(27^0)$ (h) $(216^{\frac{1}{3}})^2 \div (32^2)^{\frac{1}{5}}$ (i) $5(5^{\frac{1}{3}}) \div 5^{\frac{1}{6}}$

C

5. Simplify

(a) $(a^{16})^{\frac{1}{4}}$ (b) $(27a^3b^6)^{\frac{1}{3}}$ (c) $(125a^3)^{\frac{2}{3}}$
(d) $(32a^5)^{\frac{2}{5}}$ (e) $(81a^4)^{\frac{3}{4}}$ (f) $(8a^3b^6)^{\frac{1}{3}}$

REVIEW EXERCISE

1. Evaluate

(a) $\sqrt{144}$ (b) $\sqrt{289}$ (c) $-\sqrt{169}$ (d) $\sqrt{225}$

(e) $-\sqrt{400}$ (f) $\sqrt{441}$ (g) $\sqrt{625}$ (h) $\sqrt{576}$

2. Simplify

(a) $\sqrt{2} \times \sqrt{5} \times \sqrt{11}$ (b) $\sqrt{5} \times \sqrt{14} \times \sqrt{21}$

(c) $\sqrt{21} \times \sqrt{\frac{1}{3}}$ (d) $\sqrt{\frac{3}{5}} \times \sqrt{35}$

(e) $\sqrt{\frac{3}{4}} \times \sqrt{\frac{5}{8}}$ (f) $\sqrt{\frac{3}{8}} \times \sqrt{\frac{2}{3}}$

3. Express as mixed radicals:

(a) $\sqrt{24}$ (b) $\sqrt{54}$ (c) $\sqrt{52}$ (d) $\sqrt{98}$ (e) $\sqrt{125}$

4. Express as entire radicals:

(a) $3\sqrt{5}$ (b) $2\sqrt{7}$ (c) $6\sqrt{3}$ (d) $4\sqrt{10}$ (e) $3\sqrt{3}$

5. Simplify

(a) $\sqrt{15} + 3\sqrt{15} - 6\sqrt{15}$ (b) $\frac{1}{4}\sqrt{5} + \frac{3}{4}\sqrt{5}$

(c) $4\sqrt{27} + 2\sqrt{12}$ (d) $\sqrt{45} - \sqrt{20}$

(e) $\sqrt{98} - \sqrt{50} + \sqrt{18}$ (f) $\sqrt{28} + \sqrt{175} - \sqrt{72}$

(g) $3\sqrt{2} - 2\sqrt{8} + 2\sqrt{32}$ (h) $2\sqrt{27} + 3\sqrt{12} - \sqrt{8}$

6. Simplify

(a) $(3\sqrt{5} - \sqrt{3})(3\sqrt{5} + \sqrt{3})$ (b) $(2\sqrt{7} + \sqrt{5})(2\sqrt{7} - \sqrt{5})$

(c) $(3\sqrt{3} + 2\sqrt{5})^2$ (d) $(2\sqrt{3} - 3\sqrt{7})^2$

(e) $(\sqrt{3} + 2\sqrt{5})(3\sqrt{3} - \sqrt{5})$ (f) $(4\sqrt{7} + 2\sqrt{3})(3\sqrt{7} - 5\sqrt{3})$

(g) $(\sqrt{5} + \sqrt{3})(\sqrt{5} - \sqrt{3}) + (2\sqrt{5} + \sqrt{3})^2$

(h) $(3\sqrt{5} + 2\sqrt{2})(3\sqrt{5} - 2\sqrt{2}) + (3\sqrt{2})(\sqrt{5} + 2)$

7. Simplify by rationalizing the denominator where necessary.

(a) $\dfrac{\sqrt{45}}{\sqrt{20}}$ (b) $\dfrac{18\sqrt{24}}{6\sqrt{6}}$ (c) $\dfrac{\sqrt{7}}{\sqrt{2}}$

(d) $\dfrac{15\sqrt{24}}{3\sqrt{12}}$ (e) $\dfrac{\sqrt{18a^2b}}{\sqrt{6b}}$, $a, b > 0$ (f) $\dfrac{6\sqrt{2}}{\sqrt{6}}$

(g) $\dfrac{\sqrt{3}}{2\sqrt{10}}$ (h) $\dfrac{\sqrt{18}}{\sqrt{12}}$ (i) $\dfrac{21\sqrt{54}}{3\sqrt{6}}$

(j) $\dfrac{4}{\sqrt{11} + \sqrt{3}}$ (k) $\dfrac{3}{5 + \sqrt{3}}$ (l) $\dfrac{2\sqrt{6}}{5\sqrt{2} + \sqrt{3}}$

8. Simplify

(a) $(-1)^3(-1)^2$
(b) $3^4 \times 3 \times 3^2$
(c) $3^{10} \div 3^5$
(d) $S^5 \times S^3 \div S^4$
(e) $8^{\frac{1}{3}}$
(f) $(2^2n^3)^4$
(g) $(-1^3)^2$
(h) 10^0
(i) $(-2)^{-4}$
(j) $16^{-\frac{1}{4}}$
(k) $3^{-3} \times 3^5 \div 3$
(l) $(64a^6)^{\frac{1}{3}}$
(m) $(49m^6)^{-\frac{1}{2}}$
(n) $8^{-0.\dot{3}}$
(o) $343^{0.\dot{6}}$

9. Evaluate

(a) $10^{-5} \div 10^{-7}$
(b) $[(100^{\frac{1}{3}})^{\frac{1}{4}}]^6$
(c) $(8^{\frac{2}{3}})^2$
(d) $(10^2)^{\frac{1}{2}}$
(e) $(10^{-\frac{2}{3}})^0$
(f) $[(-1)^{\frac{1}{3}}]^6$
(g) $(12 \times 10^5) \div (3 \times 10^{-3})$
(h) $(6 \times 10^{-9}) \div (3 \times 10^{-11})$

REVIEW AND PREVIEW TO CHAPTER 7

EXERCISE 1

Express in standard notation.

1. 11.2
2. 3.75
3. 0.2575
4. 0.0325
5. 0.001 25
6. 0.000 578
7. 0.005 63
8. 0.425
9. 93 000 000
10. 186 000
11. 35 127
12. 425 000
13. 32.5
14. 3.125
15. 470.3

EXERCISE 2

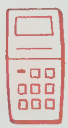
Simplify

1. (a) $10^5 \times 10^7$ (b) $10^3 \times 10^0$ (c) $10^{-5} \times 10^{-3}$
(d) 1000×10^5 (e) $10^4 \times 10^{21}$ (f) $(10^5)^3$
(g) $10^7 \div 10^4$ (h) $(10^4)^{1.5}$ (i) $(10^{1.5})^2$

2. (a) $10^4 \times 10^7$ (b) $10^{-3} \times 10^{-5}$ (c) $10^7 \times 10^{-3}$
(d) $10^7 \times 10^7$ (e) $10^{-3} \div 10^5$ (f) $10^2 \div 10^{-2}$
(g) $10^9 \div 10^7$ (h) $10^2 \div 10^{-5}$ (i) $(10^4)^3$
(j) $(10^{-3})^2$ (k) $(10^{-3})^{\frac{1}{3}}$ (l) $(10^6)^{\frac{1}{2}}$

3. (a) $10^{2+.35} \times 10^{3+.27}$ (b) $10^{4+.28} \times 10^{1+.22}$
(c) $10^{-3+.50} \times 10^{1+.25}$ (d) $10^{-3+.44} \times 10^{-2+.37}$
(e) $10^{-2+.35} \times 10^{-3+.15}$ (f) $10^{5+.85} \times 10^{2+.23}$
(g) $(10^{3+.25})^2$ (h) $(10^{-1+.37})^2$

Evaluate the following.



1. $\sqrt{1.763 + 4.817}$
2. $\sqrt{16.47 - 11.86}$
3. $\sqrt{5.813 \times 4.713}$
4. $\sqrt{21.81 \div 16.43}$
5. $\sqrt{83.65 \times 18.44 \div 12.89}$
6. $\sqrt{\dfrac{83.24 \times 6.57}{13.48}}$
7. $\sqrt{\dfrac{126.3 \times 857.4}{83.4 \times 106.7}}$
8. $\sqrt{(1.713)^2 + (8.173)^2}$
9. $\sqrt{(55.75)^3 - (16.46)^2}$
10. $\sqrt{(72.31)^3 - (19.38)^3}$

CHAPTER 7

The Exponential Function and Logarithms

A professional hockey team offered one of its players his choice of one of two bonus clauses for goals:

CLAUSE (a) $150.00 for each goal.

(b) $1.00 for the first goal, $2.00 for the second, $4.00 for the third, $8.00 for the fourth, and so on . . .

Which clause should the player accept if he expects to score

(i) 10 goals? (ii) 15 goals? (iii) 20 goals?

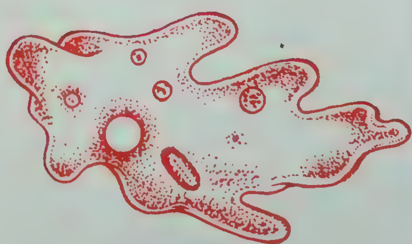
7.1 THINGS WHICH GROW EXPONENTIALLY

The main characteristic of exponential functions is the rapid way they grow in value as the value of the exponent increases. Exponential growth is demonstrated by small single-celled animals called amoebae which reproduce by splitting in half. After a period of growth they split again so that two become four. If y represents the number of amoebae and x the number of times division has taken place, the number of amoebae is given by the equation

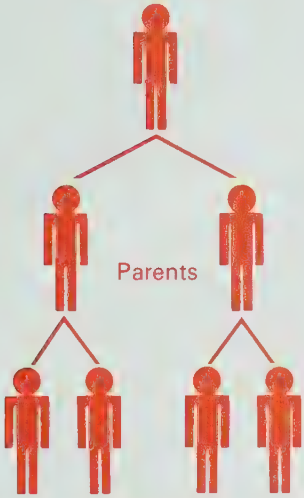
$$y = 2^x$$

This is called an exponential function since the variable x is an exponent. Under the most favourable conditions of food and temperature, the amoeba can split every 2 h. If you start with one amoeba in a bottle of water, at the end of one day it would be theoretically possible to have 2^{12} or 4096 amoebae.

$2^0 = 1$
$2^1 = 2$
$2^2 = 4$
$2^3 = 8$
$2^4 = 16$
$2^5 = 32$
$2^6 = 64$
$2^7 = 128$
$2^8 = 256$
$2^9 = 512$
$2^{10} = 1024$
$2^{11} = 2048$
$2^{12} = 4096$



EXAMPLE 1. *How many great-great-great grandparents did you have?*



Solution This represents a span of five generations. Since each individual has two parents the number of great-great-great grandparents is given by $n = 2^5$. Therefore you had 32.

EXERCISE 7-1

B

1. A civil defence warning system works on a "fan-out" phone network where the commander phones five subordinates, these five each phone five others, each of whom phone five more, etc. How many people would have been called after the fourth level had completed their calls? (Consider the commander to be the first level.)

2. A bacteria culture doubles in number every hour. If there are 11 bacteria in the original culture, how many will there be after 8 h?

3. A bacteria culture doubles in number every 90 min. If there are n bacteria in the original culture, how many will there be after 6 h?

4. The sales manager of Forest Wood Products Ltd. is in charge of sales for four departments. Each department has four area supervisors, each area supervisor has four branch managers and each branch manager has four salesmen. How many salesmen are there?

5. A radioactive material has a half-life of 2 h (ie. every 2 h the amount of material is reduced by $\frac{1}{2}$). How many units of material are left if there were 64 units 6 h ago?

6. A water vaporizer is equipped to vaporize 50% of the water it contains every hour. How long would it take to reduce 32 ℓ of water to 1 ℓ?

7. As I was going to St. Ives,
I met a man with seven wives.
Each wife had seven cats.
Each cat had seven kits.
Kits, cats, man and wives,
(a) How many were going to St. Ives?
(b) How many were coming from St. Ives?

7.2 GRAPHS OF EXPONENTIAL FUNCTIONS I

INVESTIGATION 7.2

1. Complete the table of values given below

x	$y = 2^x$	$y = 3^x$	$y = 10^x$
-2	$2^{-2} = \frac{1}{4} = 0.25$	$\frac{1}{9} \doteq 0.11$	$10^{-2} = 0.01$
-1	0.5		
0	1		
1			
2			
3			
4			

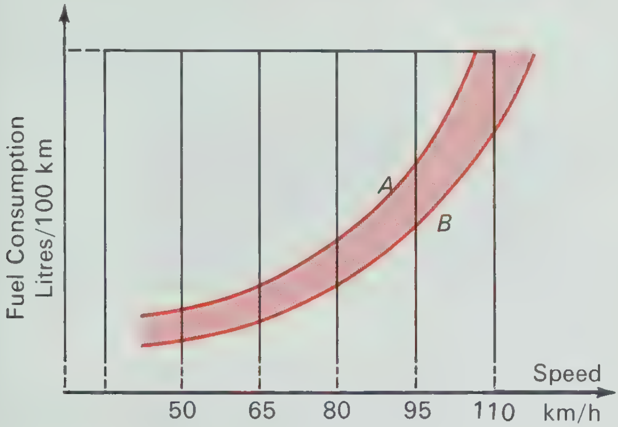
2. On a large sheet of squared paper, mark an x-axis and a y-axis. Using the same scale, mark -5 to $+5$ on the x-axis and -2 to 20 on the y-axis.

3. For each function $y = 2^x$, $y = 3^x$, $y = 10^x$ plot as many ordered pairs from the table as will fit your graph and join each set of points with a smooth curve.

4. Note in each case, that as x increases the value of y increases. Which point is common to the three graphs?

5. How does the base "a" in $y = a^x$ affect the growth of y if a is large?

A—Car carrying maximum permissible load.
B—Car occupied by two persons and carrying 350 kg of luggage.



FUEL CONSUMPTION INCREASES WITH SPEED

As the accompanying figure shows, fuel consumption does not increase uniformly with the speed, but increases *exponentially*, so that at 100 km/h a car might use almost 5 times as much gas as the same car travelling half the speed.

Other factors that can increase the number of litres per 100 km are:

1. Starting in cold weather.
2. Racing the engine.
3. Short trips in city traffic in winter.
4. Spinning the wheels.
5. Dragging at corners.
6. Low tire pressures.
7. Dirty spark plugs.
8. Engine oil too heavy.
9. Carburetor and ignition timing not properly adjusted.
10. Erratic accelerator pedal pressure.
11. Excessive use of air conditioning.

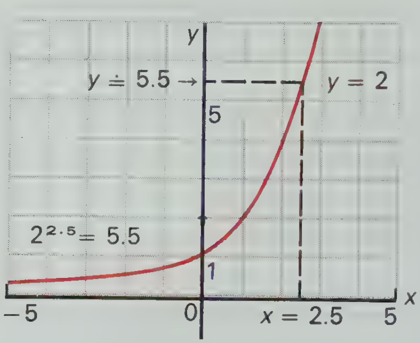
The weather can affect fuel economy to a surprising extent. For example, at highway speeds with a tailwind of 30 km/h you can average about 2.5 km more per litre than you would heading into the wind. Warm-up procedure is a major factor in fuel economy in the winter. It is wise to idle the engine a few moments then drive slowly and smoothly for the first 2 or 3 km. The shape of the car also affects fuel consumption. Aerodynamics is a factor in the design of modern cars because less power is required to overcome wind resistance if the car is "streamlined". Getting a few extra kilometres from every litre of gas depends also on good driving habits:

1. Make good use of lower speed ranges in the individual gears. 1st and 2nd gears at high speeds use a lot of gas. Shift up as soon as possible.
2. Depress the pedal gradually when accelerating. Use full pedal only in a critical traffic situation.
3. Reduce speed in good time before corners and when stopping. Do not coast down hills.
4. On the highway, accelerate smoothly to the desired speed and then ease back on the pedal to keep the car at this speed.

6. Complete the following table by taking readings from your graph.

x	$y = 2^x$	$y = 3^x$	$y = 10^x$
0.5			
1.5			
1.7			

It is simple to calculate the value of 2^2 or 3^4, but how can we find the value of $2^{2.5}$? Since $y = 2^x$ is a continuous function, it is possible to read the value from the graph.



EXERCISE 7-2

A 1. Find the value of y for each of the following, using the graph of $y = 10^x$ in the appendix.

(a) $y = 10^{0.5}$ (b) $y = 10^{0.6}$ (c) $y = 10^{0.8}$ (d) $y = 10^{1.0}$
(e) $y = 10^{0.25}$ (f) $y = 10^{0.35}$ (g) $y = 10^{0.75}$ (h) $y = 10^{0.85}$
(i) $y = 10^{0.30}$ (j) $y = 10^{0.15}$ (k) $y = 10^{0.40}$ (l) $y = 10^{0.20}$
(m) $y = 10^{0.95}$ (n) $y = 10^{0.65}$ (o) $y = 10^{0.45}$ (p) $y = 10^{0.55}$

2. Find the value of x for each of the following, using the graph of $y = 10^x$ in the appendix.

(a) $3.0 = 10^x$ (b) $6.0 = 10^x$ (c) $2.0 = 10^x$ (d) $4.0 = 10^x$
(e) $8.0 = 10^x$ (f) $3.5 = 10^x$ (g) $2.7 = 10^x$ (h) $8.5 = 10^x$
(i) $7.2 = 10^x$ (j) $9.0 = 10^x$ (k) $8.1 = 10^x$ (l) $4.7 = 10^x$
(m) $5.3 = 10^x$ (n) $6.5 = 10^x$ (o) $8.8 = 10^x$ (p) $3.7 = 10^x$

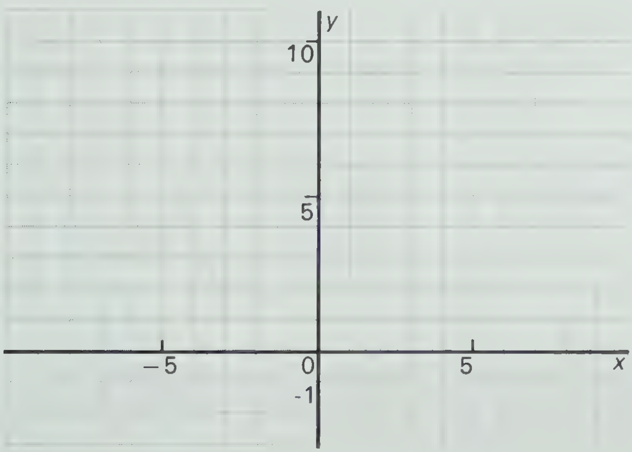
7.3 GRAPHS OF EXPONENTIAL FUNCTIONS II

INVESTIGATION 7.3

1. Complete the table of values given below.

x	$y = (\frac{1}{2})^x$	$y = (\frac{1}{3})^x$	$y = (\frac{1}{10})^x$
-2	$(\frac{1}{2})^{-2} = 2^2 = 4$	$3^2 = 9$	$10^2 = 100$
-1	2		
0	1		
1			
2			
3			
4			

2. On a large sheet of squared paper, mark an x-axis and a y-axis. Using the same scale, mark -5 to 5 on the x-axis and -1 to 10 on the y-axis.



Insert brackets to make
$3 - 2 + 1 = 5 \div 3 + 2 - 1$
a true statement.

3. For each function $y = (\frac{1}{2})^x$, $y = (\frac{1}{3})^x$, $y = (\frac{1}{10})^x$ plot as many ordered pairs from the table as will fit your graph and join each set of points with a smooth curve.

4. Note in each case that as x increases, the value of y decreases. Which point is common to the three graphs?

5. How does the base "a" in $y = a^x$ affect the decay of y if a is small?

Graphs can be used to describe the decay of radioactive metals. These metals give off radiation in the form of sub-atomic particles. This eventually results in the original metal changing to a different one. For instance, uranium gradually changes to lead. The rate at which the change takes place depends on the amount of the original material left at any given time. 1 g of radium will decay to $\frac{1}{2}$ g of radium plus other elements in 1600 a. It will take an additional 1600 a for the $\frac{1}{2}$ g to be reduced to $\frac{1}{4}$ g, and so on indefinitely. We describe this process by saying that radium has a half-life of 1600 a.

EXAMPLE 1. *How long does it take 16* g *of radium to decay to 1*g *of radium?*

Solution Let the number of half-lives required be n. Since we are reducing 16 g to 1 g, $16(\frac{1}{2})^n = 1$. Therefore $(\frac{1}{2})^n = \frac{1}{16}$.

Either

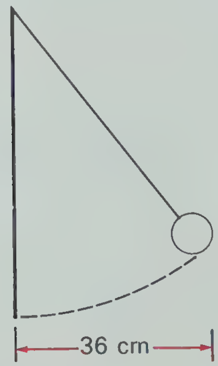
$$(\tfrac{1}{2})^4 = \tfrac{1}{16}$$
$$n = 4$$

Or

from the graph $y = 2^{-x}$

$$2^{-4} = \tfrac{1}{16}$$
$$n = 4$$

It will take four half-lives or 6400 a.



EXAMPLE 2. *A free-swinging pendulum is held 36* cm *to one side and released. Because of air resistance and friction, the bob of the pendulum returns to a position with 80% of the previous displacement. Find the amplitude after four swings.*

Solution Let the maximum displacement for each swing be A. Then:

$A_0 = 36.0$

$A_1 = 80\% \times 36.0 = (0.8)^1 \times 36.0$

$A_2 = 80\%(80\% \times 36.0) = (0.8)^2 \times 36.0$

$$A_4 = (0.8)^4(36.0) \doteq 0.410(36.0) \doteq 14.8$$

After four swings the amplitude would be 14.8 cm.

EXERCISE 7-3

1. A bouncing ball rebounds to half its previous height on each bounce. If the ball is dropped from 4 m, how high will it rebound after it has hit the ground for the fifth time?

2. If a "superball" rebounds to 90% of its previous height on each bounce and is dropped from a balcony 12 m above the sidewalk, how far will it rebound after it has hit the ground for the third time?

3. A radioactive element, actinon, has a half-life of 4 s. What fraction of an original mass of actinon is left after 20 s?

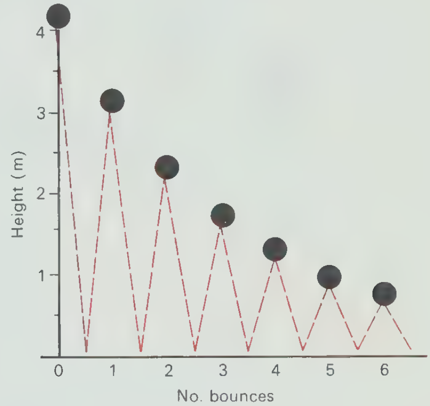
4. The accompanying graph illustrates the height to which a ball returns after being released at a height of 4 m.

(a) To what height does the ball rebound after:
   (i) the second bounce?
   (ii) the fifth bounce?

(b) What fraction of the height is lost after each bounce?

(c) How many bounces are required to reduce the height to under:
(i) 1 m (ii) 0.5 m?

(d) What percentage rebound has the ball?



5. A radioactive metal, geigerite, is found to have a half-life of 8 a. Ore containing 10% geigerite is mined and stockpiled for 16 a. Approximately what percentage of geigerite will remain?

6. To protect workers at a nuclear reactor from radiation, the radioactive material must be shielded. No practical amount of shielding will stop all the radiation. If x cm of a particular shielding material lets through half the radiation, then $2x$ cm will let through a quarter and $3x$ will let through an eighth of the radiation. If 24 cm of concrete reduces the radiation to one sixteenth of its original intensity, what thickness is required to reduce the intensity by one half?

7.4 CALCULATIONS USING THE GRAPH OF THE EXPONENTIAL FUNCTION

EXAMPLE 1 *Using the graph of $y = 10^x$ in the appendix, express 36 as a power of 10.*

Solution $36 = 3.6 \times 10$ (standard notation)

$= 10^{0.56} \times 10^1$ (read from the graph)

$= 10^{1.56}$ (add exponents)

EXAMPLE 2. *Multiply 4.8×73 using the graph of $y = 10^x$.*

Solution $4.8 \times 73 = 4.8 \times 7.3 \times 10$ (standard notation)

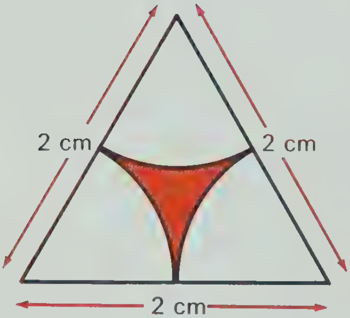
$= 10^{0.68} \times 10^{0.86} \times 10^1$ (read from graph)

$= 10^{2.54}$ (add exponents)

$= 10^{0.54} \times 10^2$ (separate fractional portion of exponent)

$= 3.5 \times 10^2$ (read from graph)

$= 3500$



Find the area of the shaded region.

EXAMPLE 3. *Evaluate* 4.7^6

Solution $4.7^6 = (10^{0.67})^6$ (read from graph)

$= 10^{4.02}$ (multiply exponents)

$= 10^{0.02} \times 10^4$

$= 1.05 \times 10^4$ (read from graph)

$= 10\ 500$

EXAMPLE 4. *Evaluate* $572 \div 26.4$.

Solution $572 \div 26.4 = (5.72 \times 10^2) \div (2.64 \times 10^1)$

$= (10^{0.76} \times 10^2) \div (10^{0.42} \times 10^1)$

$= 10^{2.76} \div 10^{1.42}$

$= 10^{1.34}$ (subtract exponents)

$= 10^{0.34} \times 10$

$= 2.2 \times 10$

$= 22$

EXERCISE 7-4

A Use the graph of $y = 10^x$ in the appendix on page.

1. Find the number represented by each of the following.

(a) $10^{0.25}$	(b) $10^{0.80}$	(c) $10^{0.75}$	(d) $10^{0.10}$
(e) $10^{0.20}$	(f) $10^{0.60}$	(g) $10^{0.40}$	(h) $10^{0.35}$
(i) $10^{0.48}$	(j) $10^{0.64}$	(k) $10^{0.62}$	(l) $10^{0.86}$

2. Find the exponent for each of the following numbers when written to base 10.

(a) 1.5	(b) 2.0	(c) 3.0	(d) 6.0
(e) 8.0	(f) 7.5	(g) 3.8	(h) 2.4
(i) 5.7	(j) 9.2	(k) 6.3	(l) 4.4

B Use the graph of $y = 10^x$ in the appendix to complete the following calculations:

3. 2.3×8.5 **4.** 3.7×4.2 **5.** $3.5 \div 2.7$ **6.** 4.7^2 **7.** 3.9×4.5

8. 46×6.3 **9.** $35 \div 6.2$ **10.** 31×48 **11.** 37^2

12. 48×730 **13.** 5600×4.7 **14.** $84 \div 13$ **15.** $510 \div 32$

16. Find the area of a rectangle 36 cm by 7.3 cm.

17. Find the volume of a cube with edge 73 cm.

18. Find the volume of a rectangular solid 25 cm by 37 cm by 52 cm.

C **19.** Find the area of a triangle with base 5.8 cm and altitude 16 cm.

20. Find the volume of a cube with edge 210 cm.

7.5 COMPUTATIONS USING EXPONENTIAL TABLES

A table of exponential values for the values of $y = 10^x$ for exponents from 0 to 1 is in the appendix. These values, having been calculated using advanced mathematical methods, are more accurate than the values that can be read from the graph in the appendix.

EXAMPLE 1. *Evaluate (a)* $10^{0.452}$ *(b)* $10^{3.476}$

Solution

(a) Read down the left-hand column to 0.45. Read across the 0.45 row to the 2 column.

$$10^{0.452} \doteq 2.831$$

Values of the Exponential Function $y = 10x$

row	0	1	2	3	4	5	6	7	8	9	Differences 1 2 3	4 5 6	7 8 9
→ 0.45	2.818	2.825	2.831	2.838	2.844	2.851	2.858	2.864	2.871	2.877	1 1 2	3 3 4	5 5 6
0.46	2.884	2.891	2.897	2.904	2.911	2.917	2.924	2.931	2.938	2.944	1 1 2	3 3 4	5 5 6
0.47	2.951	2.958	2.965	2.972	2.979	2.985	2.992	2.999	3.006	3.013	1 1 2	3 3 4	5 5 6
0.48	3.020	3.027	3.034	3.041	3.048	3.055	3.062	3.069	3.076	3.083	1 1 2	3 4 4	5 6 6
0.49	3.090	3.097	3.105	3.112	3.119	3.126	3.133	3.141	3.148	3.155	1 1 2	3 4 4	5 6 6

(b) $10^{3.476} = 10^{0.476} \times 10^3 \doteq 2.992 \times 10^3$

$$\doteq 2992$$

EXAMPLE 2. *Express as powers of 10: (a) 27.33*
(b) 0.002 956

Solution

(a) $27.33 = 2.733 \times 10^1$ Standard notation
$\doteq 10^{0.437} \times 10^1$
$\doteq 10^{1.437}$

From closest value in the body of the table.

(b) $0.002\,956 = 2.956 \times 10^{-3}$
$\doteq 10^{0.471} \times 10^{-3}$

EXAMPLE 3. *Evaluate* $10^{0.430} \times 10^{-2}$.

Solution

$$10^{0.43} \times 10^{-2} \doteq 2.692 \times 10^{-2} \quad \text{(from tables)}$$
$$= 0.026\,92$$

In example 4, we will compare the accuracy of three methods.

EXAMPLE 4. *Evaluate 572 × 46.3.*

Solution

(i) Graph of $y = 10^x$

572×46.3

$= 5.72 \times 10^2 \times 4.63 \times 10^1$

$\doteq 10^{0.76} \times 10^2 \times 10^{0.66} \times 10^1$

$\doteq 10^{4.42}$

$\doteq 10^{.42} \times 10^4$

$\doteq 2.7 \times 10^4$

$\doteq 27\,000$

(ii) Exponential table

5.72×46.3

$= 5.72 \times 10^2 \times 4.63 \times 10$

$\doteq 10^{0.757} \times 10^2 \times 10^{0.666} \times 10$

$\doteq 10^{4.423}$

$\doteq 10^{0.423} \times 10^4$

$\doteq 26\,500$

(iii) Calculator

5 7 2 x <

4 6 3 =

2 6 4 8 3 6 7

EXAMPLE 5. *Evaluate $(0.005\,63)^4$*

Solution

$(0.005\,63)^4 = (5.63 \times 10^{-3})^4$

$\doteq (10^{0.751} \times 10^{-3})^4$

$\doteq 10^{3.004} \times 10^{-12}$

$\doteq 10^{0.004} \times 10^{-9}$

$\doteq 1.009 \times 10^{-9}$

(The whole number is subtracted from the fractional exponent and added to the negative exponent)

EXAMPLE 6. *Evaluate 43.6 ÷ 925.*

Solution

Since our table contains positive exponents only, we must keep the fraction portion of the exponent positive. We do this by transferring a 1 from the exponent of 10^1 to the exponent of $10^{0.666}$, so the difference is positive.

$$\begin{aligned} 43.6 \div 925 &= (4.36 \times 10^1) \div (9.25 \times 10^2) \\ &\doteq (10^{0.640} \times 10^1) \div (10^{0.966} \times 10^2) \\ &\doteq (10^{1.640}) \div (10^{0.966} \times 10^2) \\ &\doteq 10^{0.674} \times 10^{-2} \\ &\doteq 4.72 \times 10^{-2} \\ &= 0.0472 \end{aligned}$$

Multiplication:
NORA
L
ARON

EXERCISE 7-5

$c = \pi d$
$c = 2\pi r$

A

1. Evaluate using the table of exponential values.

(a) $10^{0.315}$ (b) $10^{0.427}$ (c) $10^{0.240}$ (d) $10^{0.375}$
(e) $10^{0.928}$ (f) $10^{0.715}$ (g) $10^{0.247}$ (h) $10^{0.625}$
(i) $10^{0.427}$ (j) $10^{0.333}$ (k) $10^{0.750}$ (l) $10^{0.355}$
(m) $10^{0.855}$ (n) $10^{0.125}$ (o) $10^{0.527}$ (p) $10^{0.666}$

2. Express as powers of 10, using the exponential table.

(a) 1.995 (b) 2.259 (c) 2.438 (d) 2.594
(e) 1.099 (f) 1.107 (g) 1.403 (h) 5.715
(i) 6.871 (j) 9.886 (k) 7.603 (l) 6.471

B

3. Evaluate using the table of exponential values.

(a) $10^{2.734}$ (b) $10^{1.124}$ (c) $10^{2.563}$ (d) $10^{4.96}$
(e) $10^{1.739}$ (f) $10^{2.261}$ (g) $10^{1.37}$ (h) $10^{3.91}$
(i) $10^{0.437} \times 10^{-3}$ (j) $10^{0.436} \times 10^{-2}$ (k) $10^{0.825} \times 10^{-1}$

4. Express as powers of 10, keeping the decimal portion of the exponent positive.

(a) 275 (b) 82.4 (c) 3.597 (d) 309
(e) 5.07 (f) 1000 (g) 91 (h) 23.4
(i) 0.990 (j) 0.0641 (k) 0.000 721 (l) 0.001 27

5. Calculate the following using the table of exponential values.

(a) 47.3×2.156 (b) 248×3.721 (c) $8.73 \div 2.65$
(d) $285 \div 94.2$ (e) $(3.75)^5$ (f) $(52.4)^3$
(g) 94.2×0.0124 (h) 5.32×0.146 (i) $0.473 \div 25.4$
(j) $43.6 \div 749$ (k) $(0.047)^4$ (l) $(0.0093)^6$

6. Find the volume of a rectangular box 29.4 cm by 13.5 cm by 10.4 cm.

7. Find the circumference of a circle with diameter 19 cm ($\pi \doteq 3.14$).

8. If a ball bearing has a mass of 0.46 g, what will be the mass of a shipment of 5200?

9. If a steel casting has a mass of 375 g, how many can safely be loaded in a 5 kg crate?

10. Find the height of the flagpole in Figure 7-1 using exponential tables.

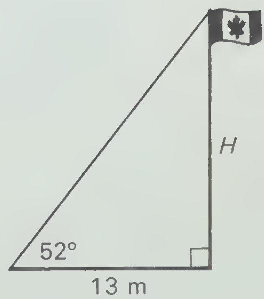


Figure 7-1
$\tan 52° \doteq 1.2799$

11. Find the area of the end of a metal rod 1.25 cm in diameter.

12. Calculate the following using the table of exponents.

(a) $0.005\,61^5$ (b) $0.073\,21 \times 58.37$
(c) $0.004\,32 \div 0.0681$ (d) $436 \div 0.0782$
(e) $574 \div 8279$ (f) $397 \div 84.6$
(g) $43.2^2 \times 825$ (h) $\dfrac{392 \times 7.24}{0.025}$
(i) $984 \div (0.086)^2$ (j) $(482 \times 3.46)^2$

Where greater accuracy is required, the exponential table is equipped with mean differences. The use of the mean difference columns is demonstrated in example 7.

EXAMPLE 7. *Evaluate (a) $10^{0.4275}$ (b) $10^{3.1423}$*

Solution

(a) $10^{0.427} \doteq 2.673$ (reading from .42 row and 7 column)

$\quad\quad\quad\quad\quad\quad 3$ (mean difference under 5 is added)

$10^{0.4275} \doteq 2.676$

Values of the Exponential Function $y = 10^x$

	0	1	2	3	4	5	6	7	8	9	Differences								
											1	2	3	4	5	6	7	8	9
0.40	2.512	2.518	2.523	2.529	2.535	2.541	2.547	2.553	2.559	2.564	1	1	2	2	3	4	4	5	5
0.41	2.570	2.576	2.582	2.588	2.594	2.600	2.606	2.612	2.618	2.624	1	1	2	2	3	4	4	5	5
0.42	2.630	2.636	2.642	2.649	2.655	2.661	2.667	2.673	2.679	2.685	1	1	2	2	3	4	4	5	6
0.43	2.692	2.698	2.704	2.710	2.716	2.723	2.729	2.735	2.742	2.748	1	1	2	3	3	4	4	5	6
0.44	2.754	2.761	2.767	2.773	2.780	2.786	2.793	2.799	2.805	2.812	1	1	2	3	3	4	4	5	6

(b) $10^{3.1423} = 10^{0.1423} \times 10^3$

$10^{0.142} \doteq 1.387$ (reading from .14 column and 2 row)

$\quad\quad\quad\quad\quad 1$ (mean difference under 3 is added)

$10^{0.1423} \doteq 1.388$

$10^{0.1423} \times 10^3 \doteq 1.388 \times 10^3$

$= 1.388$

C

13. Evaluate

(a) $10^{0.7524}$ (b) $10^{0.3748}$ (c) $10^{0.0951}$
(d) $10^{0.3545}$ (e) $10^{1.2575}$ (f) $10^{4.2107}$
(g) $10^{3.1245}$ (h) $10^{5.3344}$ (i) $10^{3.1427}$
(j) $10^{0.4275} \times 10^{-2}$ (k) $10^{0.2757} \times 10^{-1}$ (l) $10^{0.3758} \times 10^{-3}$

7.6 LOGARITHMS

power exponent

$x = 10^y$

base

The sentence $10^3 = 1000$ can also be written $3 = \log_{10} 1000$ as proposed by John Napier (1550–1617). The sentence $3 = \log_{10} 1000$ is read

"3 equals the logarithm of 1000 to base 10." This means that the logarithm of the number 1000 to base 10 is the exponent 3, to which the base must be raised to equal the number.

$$\boxed{x = 10^y \Leftrightarrow y = \log x}$$

$\log x = \log_{10} x$

When we write "log" without showing the base, the base 10 is understood. Since the logarithm of a number is the **exponent** to which 10 must be raised to give that number, logarithms will obey laws very similar to the laws of exponents.

A table of logarithms for the values of $y = \log x$ for x-values from 1 to 10 is in the appendix.

EXAMPLE 1. *Read from the tables*
(a) log 3.52 *(b) log 487* *(c) log 79 500* *(d) log 0.0257*

The logarithm table is used the same way as the exponential table.

Solution

(a) $\log 3.52 = 0.5465$

(b) $\log 487 = \log (10^2 \times 4.87)$
$= 2 + .6875$

(c) $\log 79\,500 = \log (10^4 \times 7.95)$
$= 4 + .9004$

(d) $\log 0.0257 = \log (10^{-2} \times 2.57)$
$= -2 + .4099$

Example 1 shows that a logarithm consists of two parts: an integer called the **characteristic** and a positive decimal fraction called the **mantissa.**

$$\log (\text{number}) = \boxed{\text{characteristic}} + \boxed{\text{mantissa}}$$

Note that the characteristic is always an integer and the mantissa is never negative.

Where a number has 4 significant digits, we use the mean difference columns in the tables.

EXAMPLE 2. *Read from tables:*
(a) log 5.684 *(b) log 0.002 572*

Solution

(a) $\log 5.68 \doteq 0.7543$

$\quad\quad\quad\quad\quad\quad\;\; 3 \quad$ (difference for 4)

$\log 5.684 \doteq 0.7546$

(b) $\log 0.002\,572 = \log (10^{-3} \times 2.572)$ $\quad$ $\log 2.57 \doteq 0.4099$

$= -3 + .4102$ $\quad$ difference for 2 $\quad 3$

$\log 2.572 \doteq 0.4102$

EXERCISE 7-6

A

1. Read from tables:

(a) log 5.25	(b) log 3.47	(c) log 4.28	(d) log 8.25
(e) log 7.12	(f) log 8.75	(g) log 4.95	(h) log 2.43
(i) log 2.75	(j) log 3.85	(k) log 1.17	(l) log 2.75

2. State the characteristic of each of the following:

(a) log 35.2	(b) log 456	(c) log 25	(d) log 1025
(e) log 0.0275	(f) log 0.475	(g) log 3.705	(h) log 47.5
(i) log 0.0125	(j) log 0.752	(k) log 37.05	(l) log 12 500
(m) log 7501	(n) log 357.2	(o) log 0.0045	(p) log 3.752

B

Read from tables:

3.

(a) log 5.25	(b) log 25.8	(c) log 4 250 000
(d) log 6.73	(e) log 2.58	(f) log 7.65
(g) log 45 200	(h) log 2.18	(i) log 63.7
(j) log 5280	(k) log 327 000	(l) log 7650

4.

(a) log 6.217	(b) log 6.578	(c) log 21.32
(d) log 458.7	(e) log 6 132 000	(f) log 405 300
(g) log 42.16	(h) log 23.17	(i) log 30 250
(j) log 2.858	(k) log 3 257 000	(l) log 37.54

5.

(a) log 2.575	(b) log 0.4844	(c) log 0.075 72
(d) log 0.004 285	(e) log 13.27	(f) log 0.004 215
(g) log 118.5	(h) log 0.1185	(i) log 1.185

7.7 MULTIPLICATION USING LOGARITHMS

$$\boxed{\log AB = \log A + \log B}$$

$A = 10^x,\ B = 10^y$
$\log A = x$
$\log B = y$
$\log AB$
$= \log(10^x \times 10^y)$
$= \log 10^{x+y}$
$= x + y$
$= \log A + \log B$

This law can be checked using examples as follows:

$$\begin{aligned}\log (7 \times 2) &= \log 14\\ &= \log (10^1 \times 1.4)\\ &\doteq 1 + .1461\\ &\doteq 1.1461\\ \log 7 + \log 2 &\doteq 0.8451 + 0.3010\\ &\doteq 1.1461\end{aligned}$$

$\therefore \log(7 \times 2) = \log 7 + \log 2$

Check the law with other examples.

EXAMPLE 1. *Evaluate 568 × 58 using logarithms.*

Solution

$$\begin{aligned}
\text{Let } x &= 568 \times 58 \\
\log x &= \log(568 \times 58) \\
&= \log 568 + \log 58 \\
&\doteq (2 + .7543) + (1 + .7634) \\
&\doteq 3 + 1.5177 \\
&\doteq 4 + .5177 \\
x &\doteq 10^{4 + .5177} \\
&\doteq 10^4 \times 3.294 \\
568 \times 58 &\doteq 3.294 \times 10^4
\end{aligned}$$

$$\begin{array}{r}
\log 568 = 2 + .7543 \\
\log 58 = 1 + .7634 \\
\hline
3 + 1.5177
\end{array}$$

EXERCISE 7-7

A

1. Express as sums.

(a) log (25 × 35) (b) log (75 × 15)
(c) log (35 × 17) (d) log 25^2
(e) log (15.2 × 3.54) (f) log (3125 × 41.27)
(g) log (4.27 × 5.83) (h) log (42.7 × 10.2)
(i) log (325 × 1.4) (j) log (38.51 × 21.7)

B

2. Add the following logarithms.

(a) $\begin{array}{r} 1 + .3752 \\ 4 + .2615 \\ \hline \end{array}$ (b) $\begin{array}{r} 4 + .2575 \\ -2 + .1053 \\ \hline \end{array}$ (c) $\begin{array}{r} 5 + .7215 \\ -3 + .4117 \\ \hline \end{array}$

(d) $\begin{array}{r} -3 + .5728 \\ -2 + .3755 \\ \hline \end{array}$ (e) $\begin{array}{r} -3 + .1157 \\ -4 + .4725 \\ \hline \end{array}$ (f) $\begin{array}{r} -5 + .2470 \\ 5 + .9123 \\ \hline \end{array}$

(g) $\begin{array}{r} -3 + .2157 \\ 4 + .3148 \\ 6 + .1527 \\ \hline \end{array}$ (h) $\begin{array}{r} 5 + .2517 \\ -3 + .2114 \\ -5 + .7205 \\ \hline \end{array}$ (i) $\begin{array}{r} -4 + .7835 \\ -2 + .4721 \\ 3 + .9847 \\ \hline \end{array}$

3. Evaluate using logarithms.

(a) 5.25 × 3.75 (b) 6.35 × 78.2 (c) 2.34 × 0.136
(d) 78.4 × 2.16 (e) 6.27 × 8.41 (f) 355 × 21.7
(g) 5.85 × 0.125 (h) 41.3 × 18.2 (i) 265 × 2.45

4. Evaluate using logarithms.

(a) 54 700 × 3720 (b) 32.8 × 47.2 (c) 0.0557 × 0.352
(d) 40 × 50 (e) 356 × 47.3 (f) 0.0055 × 0.253
(g) 0.005 29 × 3100 (h) 478 × 0.005 25 (i) 3.15 × 48.8

5. Evaluate using logarithms:

(a) 10.1 × 14.1 × 0.004 34 (b) 3.66 × 4.84 × 4.62
(c) 2.648 × 7.53 × 16.25 (d) 17.86 × 5.937 × 6.527
(e) 4.17 × 4.36 × 0.673 (f) 0.5647 × 0.3835 × 6.82

6. Find the area of a rectangular factory 86.4 m by 65.7 m.

7. Find the volume of a box 0.4770 m by 0.3421 m by 2.468 m.

8. The density, D, of a certain block is 843.5 kg/m³. Find the mass, m of the block if its volume, V, is 6.594×10^{-6} m^3 ($m = D \times V$).

9. Find the volume of a cube with each side 3.125 m.

10. If 0.3275 g of material are deposited at the anode in 1 s, how many grams are deposited in 4.375 s?

7.8 DIVISION USING LOGARITHMS

$$\boxed{\log \frac{A}{B} = \log A - \log B}$$

$A = 10^x$, $B = 10^y$
$\log A = x$
$\log B = y$

$$\log \frac{A}{B} = \log \frac{10^x}{10^y}$$
$$= \log 10^{x-y}$$
$$= x - y$$
$$= \log A - \log B$$

This law can be checked using examples as follows

$$\log (6 \div 2) = \log 3$$
$$\doteq 0 + .4771$$
$$\log 6 - \log 2 \doteq (0 + .7782) - (0 + .3010)$$
$$\doteq 0 + .4772$$

The discrepancy of 1 in the fourth decimal place is due to rounding off in the preparation of tables. Check the law with other examples.

EXAMPLE 1. *Evaluate 234.6 ÷ 7.408 using logarithms.*

Solution

$$\text{Let } x = 234.6 \div 7.408$$
$$\log x = \log 234.6 - \log 7.408$$
$$\log 234.6 = 2 + .3703$$
$$\log 7.408 = \underline{0 + .8697}$$
$$\log x = 1 + .5006$$
$$x = 10^{1\ +\ .5006}$$
$$= 10^1 \times 3.166$$
$$234.6 \div 7.408 = 31.66$$

$$\overset{1}{\cancel{2}} + 1.3703$$
$$0 + .8697$$

EXAMPLE 2. *Evaluate 0.006 523 ÷ 0.000 216 3 using logarithms.*

Solution

$$\text{Let } x = 0.006\,523 \div 0.000\,216\,3$$
$$\log x = \log 0.006\,523 - \log 0.000\,216\,3$$
$$\log 0.006\,523 = -3 + .8144$$
$$\log 0.000\,216\,3 = \underline{-4 + .3351}$$
$$\log x = 1 + .4793$$
$$x = 10^{1 + .4793}$$
$$= 10^1 \times 3.015$$
$$0.006\,523 \div 0.000\,216\,3 \doteq 30.15$$

EXERCISE 7-8

Division:

$$\begin{array}{r} SIR \\ BE\,\overline{)ABLE} \\ \underline{MR} \\ RRL \\ \underline{RLM} \\ BE \\ \underline{BE} \end{array}$$

A

1. Express as differences.

(a) $\log (30 \div 7)$ (b) $\log (40 \div 9)$
(c) $\log (1 \div 5)$ (d) $\log (7.5 \div 7.5)$
(e) $\log \frac{47}{35}$ (f) $\log \frac{85}{126}$
(g) $\log \frac{3.6}{12.8}$ (h) $\log \frac{52.7}{12.3}$
(i) $\log (0.5 \div 0.075)$ (j) $\log (31.26 \div 54.85)$

2. Subtract the following logarithms.

(a) $\begin{array}{r} 4 + .5734 \\ \underline{2 + .2518} \end{array}$ (b) $\begin{array}{r} 5 + .8437 \\ \underline{-2 + .4127} \end{array}$ (c) $\begin{array}{r} 6 + .7205 \\ \underline{-3 + .1816} \end{array}$

(d) $\begin{array}{r} -3 + .6724 \\ \underline{-2 + .2517} \end{array}$ (e) $\begin{array}{r} -4 + .2508 \\ \underline{-6 + .1247} \end{array}$ (f) $\begin{array}{r} -4 + .2740 \\ \underline{4 + .8125} \end{array}$

(g) $\begin{array}{r} 3 + .2157 \\ \underline{5 + .8417} \end{array}$ (h) $\begin{array}{r} -4 + .3116 \\ \underline{-2 + .4205} \end{array}$ (i) $\begin{array}{r} 0 + .2507 \\ \underline{-3 + .5012} \end{array}$

B

3. Evaluate using logarithms.

(a) $358 \div 26$ (b) $4570 \div 257$ (c) $4890 \div 2710$
(d) $58.3 \div 35.1$ (e) $625 \div 4.8$ (f) $3.14 \div 2.17$
(g) $68.9 \div 25.3$ (h) $3.43 \div 15.7$ (i) $0.005 \div 0.000\,74$

4. Evaluate using logarithms.

(a) $4.375 \div 2.557$ (b) $14.68 \div 25.35$
(c) $137.8 \div 254.7$ (d) $16.66 \div 9.333$
(e) $0.005\,741 \div 0.2163$ (f) $3.755 \div 5.858$
(g) $0.7225 \div 6.735$ (h) $0.021\,71 \div 0.001\,134$
(i) $358.2 \div 0.023\,75$

C

5. Evaluate using logarithms.

(a) $3.25 \times 2.46 \div 4.65$ (b) $427 \times 318 \div 675$
(c) $0.527 \times 0.643 \div 8.25$ (d) $32.5 \times 0.735 \div 11.2$
(e) $55.7 \times 21.6 \div 312$ (f) $0.0255 \times 2.17 \div 0.0375$

$$\log \frac{AB}{C}$$
$$= \log A + \log B - \log C$$

$$\log \frac{A}{BC} = ?$$

6. Evaluate using logarithms.

(a) $35.8 \div (21.5 \times 5.12)$ (b) $64.7 \div (55.3 \times 0.214)$
(c) $0.0575 \div (0.125 \times 3.14)$ (d) $5.75 \div (0.0525 \times 0.0388)$

7. Evaluate using logarithms.

(a) $\dfrac{3.352 \times 51.74}{3.571}$ (b) $\dfrac{1}{6.286 \times 4.724}$ (c) $\dfrac{0.583}{7.185 \times 288.4}$

(d) $\dfrac{39.7 \times 6.92}{4.96 \times 849}$ (e) $\dfrac{91 \times 23.4}{86.4 \times 18.2}$ (f) $\dfrac{0.3621 \times 891.3}{0.04213 \times 32.63}$

8. Find R if $V = 23.46$ and $I = 7.408$, where $R = \dfrac{V}{I}$.

9. Find the density, D, of 0.000 019 7 kg of a substance which occupies $4.59 \times 10^{-4}\,m^3$, where $D = \dfrac{m}{V}$.

$$\begin{aligned} A &= 10^x \\ x &= \log A \\ \log A^n &= \log (10^x)^n \\ &= \log 10^{nx} \\ &= nx \\ &= n \log A \end{aligned}$$

7.9 POWERS AND ROOTS USING LOGARITHMS

$$\boxed{\log A^n = n \log A}$$

This law can be checked using examples as follows:

$$\begin{aligned} \log 2^3 &= \log 8 \\ &\doteq 0 + .9031 \\ 3 \log 2 &\doteq 3(0 + .3010) \\ &\doteq 0 + .9030 \end{aligned}$$

Explain why there is a discrepancy of 1 in the fourth decimal place. Check this law with other examples.

EXAMPLE 1. *Evaluate* $(1.05)^{21}$

Solution

$$\begin{aligned} \text{Let } x &= 1.05^{21} \\ \log x &= \log 1.05^{21} \\ &= 21 \log 1.05 \\ &\doteq 21(0 + .0212) \\ &\doteq 0 + .4450 \\ x &\doteq 10^{0+.4450} \\ &\doteq 2.786 \\ 1.05^{21} &\doteq 2.786 \end{aligned}$$

EXAMPLE 2. *Evaluate (a)* $\sqrt{69.23}$ *(b)* $\sqrt[7]{0.063\,98}$

Solution

(a)

$$\begin{aligned} \text{Let } x &= \sqrt{69.23} = (69.23)^{\frac{1}{2}} \\ \log x &= \log (69.23)^{\frac{1}{2}} \\ &\doteq \tfrac{1}{2} \log 69.23 \\ &\doteq \tfrac{1}{2}(1 + .8403) \\ &\doteq \tfrac{1}{2}(1.8403) \\ &\doteq 0.9202 \\ &\doteq 0 + .9202 \\ x &\doteq 10^{0+.9202} \\ &\doteq 8.322 \\ \sqrt{69.23} &\doteq 8.322 \end{aligned}$$

(b)

$$\begin{aligned} \text{let } x &= \sqrt[7]{0.063\,98} \\ &= (0.063\,98)^{\frac{1}{7}} \\ \log x &= \log (0.063\,98)^{\frac{1}{7}} \\ &= \tfrac{1}{7} \log 0.063\,98 \\ &\doteq \tfrac{1}{7}(-2 + .8060) \\ &\doteq \tfrac{1}{7}(-7 + 5 + .8060) \\ &\doteq \tfrac{1}{7}(-7 + 5.8060) \\ &\doteq -1 + .8294 \\ x &\doteq 10^{-1+.8294} \\ &\doteq 10^{-1} \times 6.752 \\ \sqrt[7]{0.063\,98} &\doteq 6.752 \times 10^{-1} \\ &\doteq 0.6752 \end{aligned}$$

$$\begin{aligned} &-2 + 0.8 \\ &= -2 - 5 + 5 + 0.8 \\ &= -7 + 5.8 \end{aligned}$$

EXERCISE 7-9

A **1.** Apply the power law to the following:

(a) $\log 5.5^2$ (b) $\log 3.75^3$ (c) $\log 25.5^4$
(d) $\log 425^5$ (e) $\log 2.353^{\frac{1}{2}}$ (f) $\log 2.47^{\frac{1}{4}}$
(g) $\log \sqrt{375}$ (h) $\log \sqrt[3]{21.65}$ (i) $\log \sqrt[5]{3.125}$
(j) $\log \sqrt[4]{0.275}$

B **2.** Simplify

(a) $3(2 + .4125)$ (b) $3(-3 + .5012)$ (c) $\frac{1}{2}(2 + .3175)$
(d) $\frac{1}{3}(-3 + .4265)$ (e) $\frac{1}{4}(-3 + .4265)$ (f) $\frac{1}{5}(-2 + .2116)$
(g) $\frac{1}{2}(-3 + .2117)$ (h) $\frac{1}{3}(-2 + .4127)$ (i) $\frac{1}{6}(-1 + .2854)$

3. Evaluate using logarithms.

(a) 256^2 (b) 2.75^4 (c) 3.78^5
(d) 5.27^3 (e) 21.8^2 (f) 3.12^4
(g) 0.375^4 (h) 0.0275^4 (i) 0.138^3

4. Evaluate using logarithms.

(a) $125^{\frac{1}{2}}$ (b) $1030^{\frac{1}{3}}$ (c) $2.55^{\frac{1}{2}}$
(d) $\sqrt{586}$ (e) $\sqrt{2140}$ (f) $\sqrt[3]{31.2}$
(g) $\sqrt{0.425}$ (h) $\sqrt[3]{0.002\,57}$ (i) $\sqrt[3]{0.003\,15}$

5. Evaluate using logarithms.

(a) 35.25^3 (b) 1.148^2 (c) 2.735^5
(d) $\sqrt{528.4}$ (e) $\sqrt[3]{130.4}$ (f) $\sqrt{257.5}$
(g) $0.037\,75^3$ (h) $\sqrt{0.2175}$ (i) $\sqrt[3]{0.084\,75}$

C **6.** Evaluate using logarithms.

(a) 3.14×1.85^2 (b) 40.1×5.27^2
(c) 3.14×1.75^2 (d) 5.27×0.375^3
(e) 3.142×0.2785^2 (f) $42.65^2 \times 21.7$

7. Interest on \$1000 at 10%/a, compounded semi-annually for 10 a is given by the expression

$$1000\,(1.05)^{20}$$

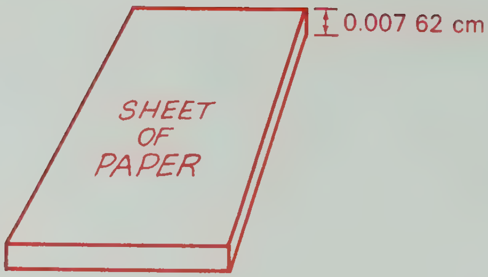
Find the interest to the nearest dollar.

8. Calculate the area of a circle with radius 2.12 cm to two decimal places.

9. Find the volume of a spherical ball-bearing with radius 0.470 cm.

$V = \frac{4}{3}\pi r^3$

10. The period, T, of a pendulum is given by $T = 2\pi\sqrt{\frac{l}{g}}$ where T is in seconds. Find T if $\pi \doteq 3.142$, $l = 125.5$ cm, and $g = 974.2$ cm/s^2.



A sheet of paper was measured with a micrometer and found to be 0.007 62 cm thick. If the paper were folded 50 times, how thick is the stack of paper?

Folds	Thickness
0	0.007 62
1	$0.007\,62 \times 2$
2	$0.007\,62 \times 4 = 0.007\,62 \times 2^2$
3	$0.007\,62 \times 8 = 0.007\,62 \times 2^3$
4	$0.007\,62 \times 16 = 0.007\,62 \times 2^4$
5	$0.007\,62 \times 32 = 0.007\,62 \times 2^5$
⋮	⋮
50	$0.007\,62 \times 2^{50}$

There are $100 \times 1000 \text{ cm} = 10^5 \text{ cm}$ in one kilometre. The thickness of the stack of paper is

$$\frac{0.007\,62 \times 2^{50}}{10^5} \text{ km}$$

Evaluate this number.

REVIEW EXERCISE

B

1. On the same axes, sketch the following graphs:

(a) $y = 2^x \quad , \quad -4 \leq x \leq 4$

(b) $y = 3^x \quad , \quad -3 \leq x \leq 3$

(c) $y = 4^x \quad , \quad -2 \leq x \leq 2$

2. On the same axes sketch the following graphs:

(a) $y = 2^x \quad , \quad -4 \leq x \leq 4$

(b) $y = (\frac{1}{2})^x \quad , \quad -4 \leq x \leq 4$

3. On the same axes sketch the following graphs:

(a) $y = (\frac{1}{2})^x \quad , \quad -4 \leq x \leq 4$

(b) $y = (\frac{1}{3})^x \quad , \quad -3 \leq x \leq 3$

(c) $y = (\frac{1}{4})^x \quad , \quad -2 \leq x \leq 2$

4. Evaluate using a graph or tables.

(a) $10^{0.25}$ (b) $10^{0.75}$ (c) $10^{0.50}$ (d) $10^{0.40}$
(e) $10^{2.15}$ (f) $10^{3.75}$ (g) $10^{2.14}$ (h) $10^{0.33}$

5. Express as powers of ten.

(a) 3.25 (b) 5.27 (c) 8.14 (d) 7.34
(e) 0.75 (f) 2.17 (g) 5.92 (h) 4.33

6. Evaluate using logarithms.

(a) 3.25×4.75 (b) 58.7×21.6 (c) 0.258×3.14
(d) $38.4 \div 2.75$ (e) $0.754 \div 2.17$ (f) $35.3 \div 0.273$
(g) 52.5^3 (h) 0.275^4 (i) $3.75^{\frac{1}{2}}$
(j) $\sqrt{58.6}$ (k) $\sqrt[3]{0.005\,79}$ (l) $\sqrt[4]{0.213}$

C **7.** Evaluate using logarithms.

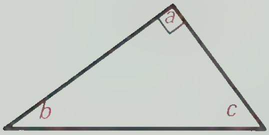
(a) 3.14×2.15^2 (b) $55.2 \div (3.75 \times 2.45)$
(c) $\dfrac{5.255 \times 2.475}{3.751 \times 21.65}$ (d) $\dfrac{(52.75)^2}{(38.75)(41.26)}$

8. Evaluate using logarithms:

(a) $\dfrac{\sqrt{3.125}}{5.265 \times 21.83}$ (b) $\sqrt{\dfrac{35.85 \times 0.2175}{857.2}}$

(c) $\dfrac{(35.65)^2}{\sqrt{685.3}}$ (d) $\dfrac{84.75}{\sqrt{35.61 \times 124.6}}$

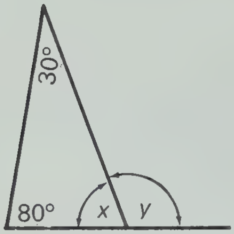
REVIEW AND PREVIEW TO CHAPTER 8



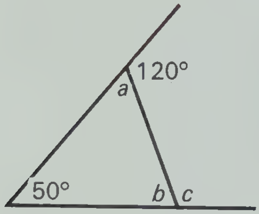
$a + b + c = 180°$

1. Calculate the indicated angles.

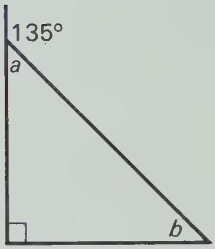
(a)



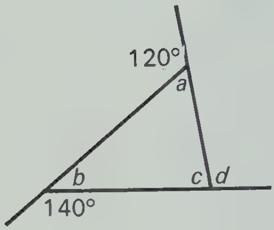
(b)



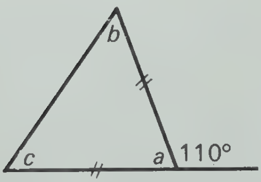
(c)



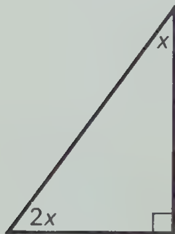
(d)



(e)



(f)



2. Find the value of x in the following.

$\frac{a}{b} = \frac{c}{d}$

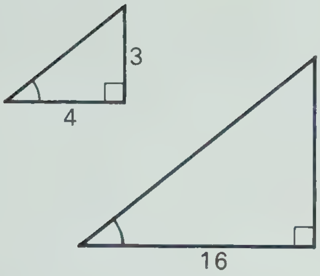
$ad = bc$

(a) $\frac{x}{7} = \frac{12}{14}$ (b) $\frac{x}{12} = \frac{3}{4}$ (c) $\frac{5}{x} = \frac{15}{36}$

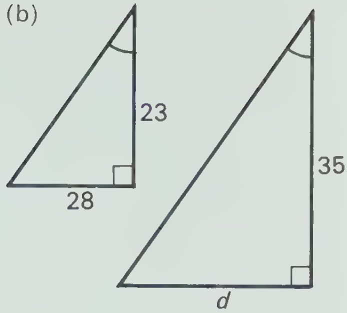
(d) $\frac{5}{x} = \frac{15}{12}$ (e) $\frac{5}{13} = \frac{10}{x}$ (f) $\frac{7}{28} = \frac{x}{16}$

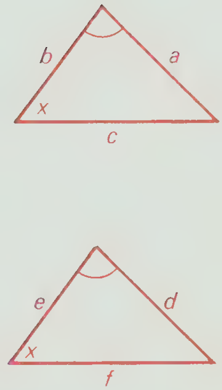
3. Compute the indicated dimensions in each of the following pairs of triangles.

(a)



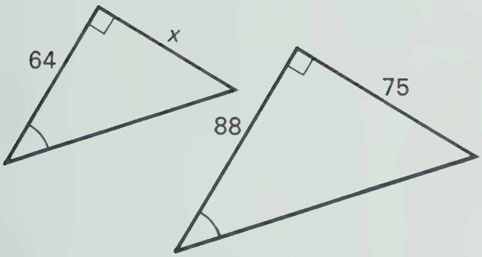
(b)



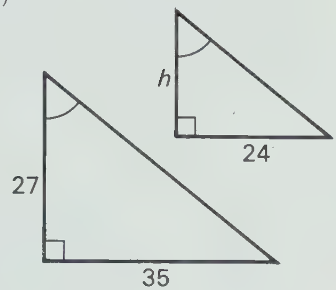


$\frac{a}{d} = \frac{b}{e} = ?$

(c)



(d)



4. A flagpole casts a 16 m shadow at the same time that a man 2 m tall casts a 1.5 m shadow. Make a diagram showing the two similar triangles and calculate the height of the flagpole.

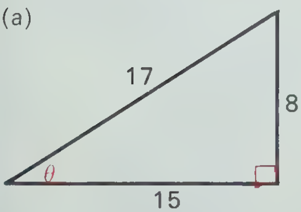
5. State the value of (i) $\sin\theta$, (ii) $\cos\theta$, (iii) $\tan\theta$ in each of the following.

$\sin\theta = \frac{opp}{hyp}$

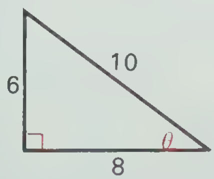
$\cos\theta = \frac{adj}{hyp}$

$\tan\theta = \frac{opp}{adj}$

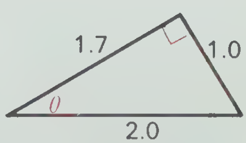
(a)



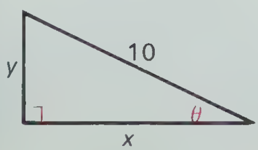
(b)



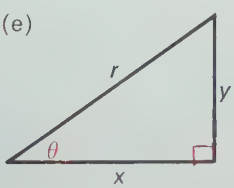
(c)



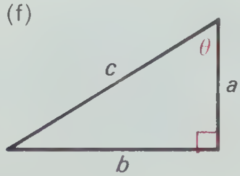
(d)



(e)

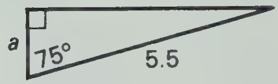


(f)

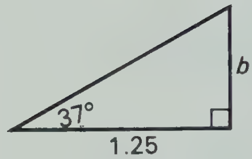


6. Find the length of the indicated side:

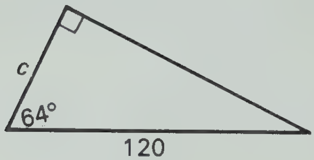
(a)



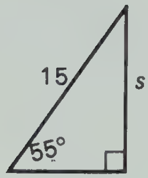
(b)



(c)

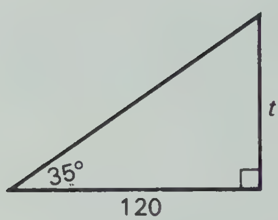


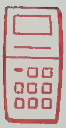
(d)



(e)

(f)





Solve for *x*.

1. $\dfrac{x}{13.76} = \dfrac{18.51}{15.76}$

2. $\dfrac{x}{57.29} = \dfrac{48.76}{33.54}$

3. $\dfrac{58.72}{72.83} = \dfrac{x}{2.67}$

4. $\dfrac{582.7}{633.5} = \dfrac{x}{104.6}$

5. $\dfrac{85.76}{x} = \dfrac{95.43}{18.71}$

6. $\dfrac{4.743}{x} = \dfrac{5.894}{6.382}$

7. $\dfrac{21.73}{15.46} = \dfrac{77.81}{x}$

8. $\dfrac{0.7134}{0.8123} = \dfrac{0.5812}{x}$

9. $\dfrac{x}{3.712} = \dfrac{3.712}{4.813}$

10. $\dfrac{x}{157.4} = \dfrac{157.4}{481.6}$

THEODOLITES AND TRIGONOMETRY

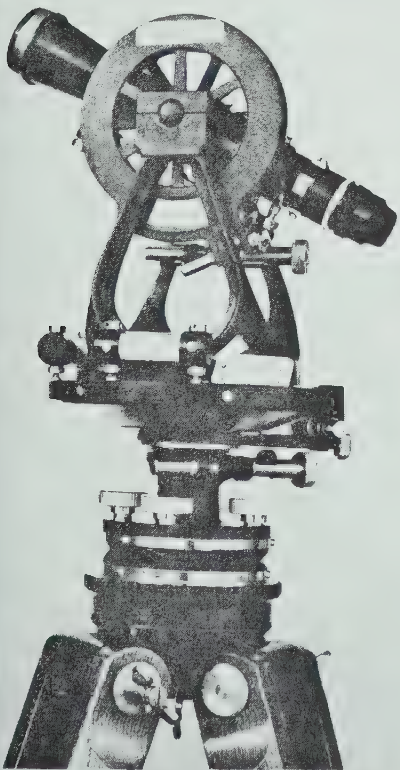
A *theodolite* is an instrument for measuring horizontal and vertical angles. It is extensively used by civil engineers and surveyors to calculate distances when it is too difficult or impossible to make linear measurements.

A telescope, giving the line of sight, is mounted on a horizontal axis, which is in turn fixed on a base which can swing through 360°. Modern theodolites are of the *transit* type, illustrated here, in which the telescope can also make a complete rotation about the horizontal axis (in the vertical plane), reversing the line of sight. Vertical angles measured upward from the horizontal are called *angles of elevation*, those measured downward are *angles of depression.* Theodolites have plumb lines or other gravity levelling devices, sometimes automatic, so that the base on the tripod can be set absolutely horizontal no matter how uneven the ground.

To measure the height of a flagpole, for instance, a surveyor would set up his transit a known distance from the base of the pole, and sight the top of it through the telescope. With this one angle and one side of a right angle triangle (assuming the pole is vertical), he could then calculate the height of the pole, using trigonometric ratios. A similar method could be used to find the width of a river.

A transit might also be used to make a diagram of a piece of land by measuring its sides and determining the angles between them.

Preliminary survey work is often done with a smaller instrument like the three-inch diameter pocket transit illustrated at the bottom of this page. It will determine horizontal and vertical angles, and can be used as a prismatic compass level, a clinometer, a plumb, or an alidade.



Courtesy Hughes-Owens

Courtesy Hughes-Owens

CHAPTER 8

Trigonometry

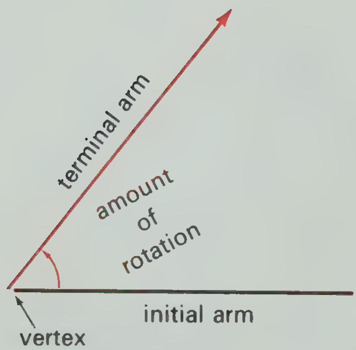
Trigonometry is a branch of mathematics used more than 2000 years ago by astronomers, navigators, and surveyors to determine distances to inaccessible objects, and to determine unmeasurable heights. Today, trigonometry continues to grow in importance not only in the space program but also in the day-to-day work of skilled machinists, draftsmen, and other modern-day technologists.

8.1 ANGLES IN STANDARD POSITION

An angle is in standard position if it is related to a pair of coordinate axes so that its vertex is at the origin and the initial arm lies along the positive x-axis as in Figure 8-1. If P is a point on the terminal arm, then $\angle POX$ is in standard position. The distance OP is found using the Pythagorean theorem.

$$OP^2 = 3^2 + 4^2 = 25$$
$$OP = 5$$

The length OP is always positive.



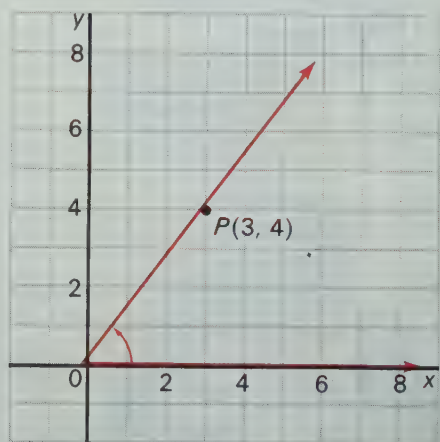


Figure 8-1

EXERCISE 8-1

1. (a) Draw a diagram showing point A (6,8) on the terminal arm of $\angle AOX$ in standard position.
(b) Calculate the length OA.

2. In Figure 8-2, $\angle BOX$, $\angle COX$, and $\angle DOX$ are angles in standard position. Calculate the lengths of:

(a) OB (b) OC (c) OD

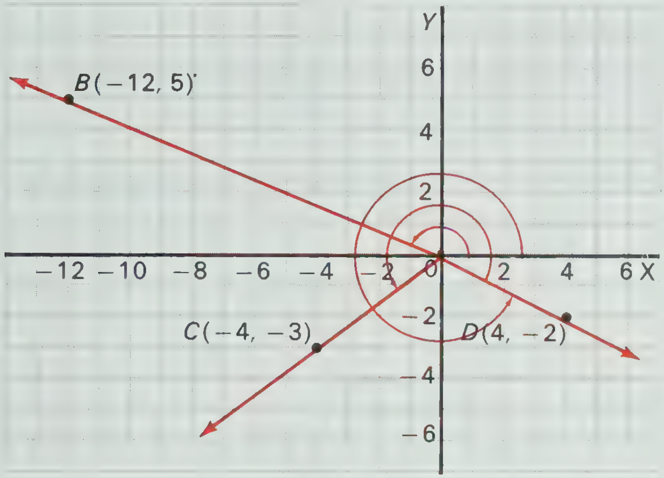


Figure 8-2

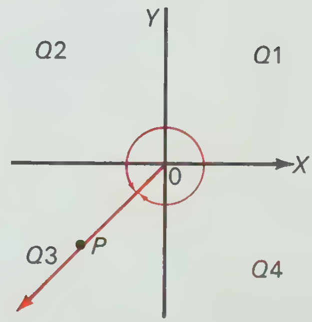


Figure 8-3

3. In Figure 8-3, the coordinate axes divide the graph into four quadrants. The angle in standard position, $\angle POX$, is a third quadrant angle, because the terminal arm OP lies in the third quadrant. An angle which is the result of counterclockwise rotation is called a positive angle. A negative angle is the result of a clockwise rotation.
(a) Draw a diagram of a second quadrant angle. Mark the arrow of positive rotation.
(b) Draw a diagram of a fourth quadrant angle. Mark the arrow of negative rotation.

4. A rotation can be zero, part of a revolution, a complete revolution, or more than a revolution.

One complete revolution has a measure of 360°.

Draw each of the following angles in standard position and show the arrow of rotation.

(a) +30°	(b) +45°	(c) −60°
(d) 135°	(e) −225°	(f) 270°
(g) 315°	(h) −360°	(i) 460°

Note: When a *variable* represents the measure of an angle, the degree symbol can be omitted.

Test your skill:
0.0002×16

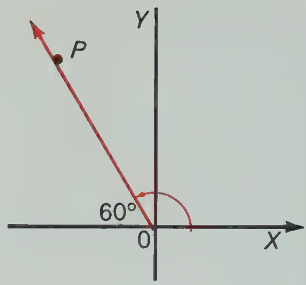
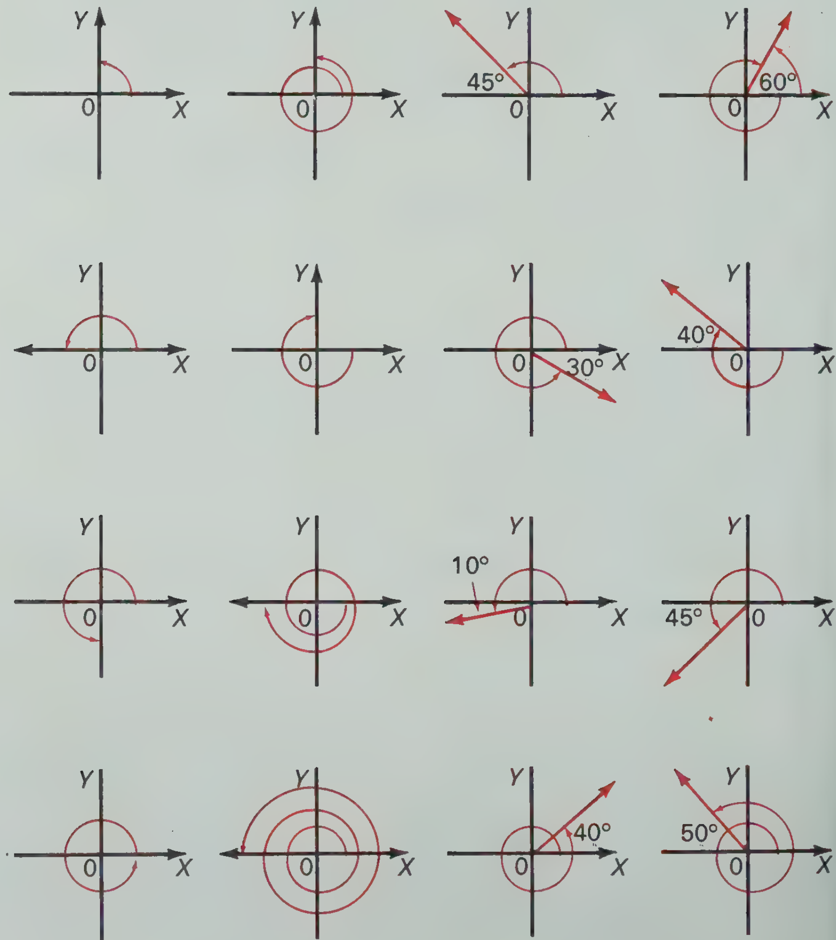


Figure 8-4

5. $\angle POX$ (Figure 8-4) is in standard position.
(a) State the measure of $\angle POX$.
(b) State the measure of three other angles having the same terminal arm as a result of positive rotation.
(c) State the measure of three other angles having the same terminal arm as the result of negative rotation.

6. State the measurement in degrees of each of the angles indicated:



8.2 SINE, COSINE, AND TANGENT

In Figure 8-5, $P(x,y)$ is any point on the terminal arm of $\angle POX$ in standard position. The length, r, from the origin to the point $P(x,y)$, is found using

$$\boxed{r = \sqrt{x^2 + y^2}}$$

Note: While x and y may be positive or negative, r is always positive.

For an angle θ in standard position, the primary trigonometric function values, sine θ, cosine θ, and tangent θ, are:

$$\boxed{\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad \tan\theta = \frac{y}{x},\ x \neq 0}$$

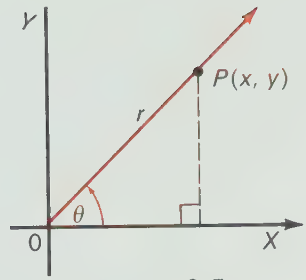


Figure 8-5

Since the trigonometric functions are defined by means of fractions, the trigonometric function values are often called ratios.

EXAMPLE 1. *The point (1, 5) lies on the terminal arm of $\angle\theta$ in standard position. Find the primary trigonometric ratios.*

Solution

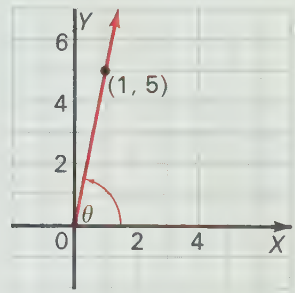
$x = 1, \quad y = 5$

$r = \sqrt{x^2 + y^2}$

$\therefore r = \sqrt{1^2 + 5^2}$

$= \sqrt{26}$

$\sin\theta = \dfrac{5}{\sqrt{26}}, \quad \cos\theta = \dfrac{1}{\sqrt{26}}, \quad \tan\theta = \dfrac{5}{1} = 5.$



EXAMPLE 2. *If θ is a second quadrant angle and $\tan\theta = -\frac{4}{5}$, find $\sin\theta$ and $\cos\theta$.*

Solution

$\tan\theta = -\frac{4}{5}$. Since θ is in the second quadrant, $y = 4$ and $x = -5$ are one pair of values for x and y.

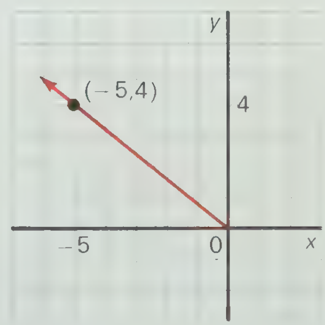
$r = \sqrt{x^2 + y^2}$

$r = \sqrt{(-5)^2 + (4)^2}$

$= \sqrt{25 + 16}$

$= \sqrt{41}$

$\sin\theta = \dfrac{4}{\sqrt{41}}$ and $\cos\theta = \dfrac{5}{\sqrt{41}}$



EXAMPLE 3. *If $\cos\theta = \dfrac{5}{\sqrt{34}}$:*

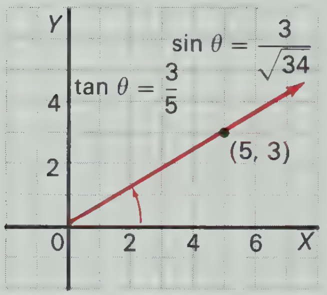
(a) Find all possible values of y for a value of x.
(b) Sketch the angles on a pair of axes.
(c) Indicate the values of $\sin\theta$ and $\tan\theta$ for each possibility.

Solution

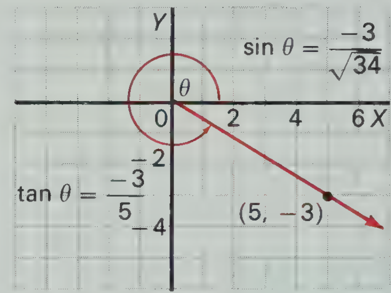
$\cos\theta = \dfrac{5}{\sqrt{34}}$; then $x = 5$ and $r = \sqrt{34}$ is one set of values.

$$x^2 + y^2 = r^2$$
$$25 + y^2 = 34$$
$$y^2 = 9$$
$$y = +3 \text{ or } y = -3$$

For $y = +3$:



For $y = -3$:



EXERCISE 8-2

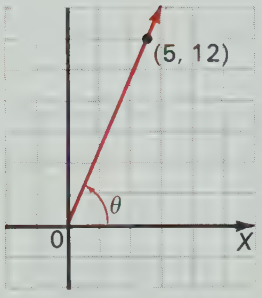
A

1. For each of the diagrams below:
   (a) State the value of r.
   (b) State the primary trigonometric ratios.

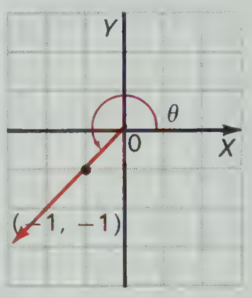
If all the letters represent natural numbers, and $C > 2$, solve the cryptogram:

$$\frac{PORK}{CHOP} = C$$

(i)



(ii)



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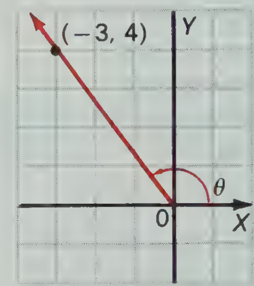
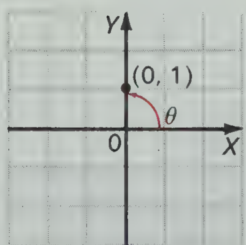


(iv)



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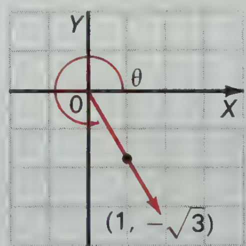

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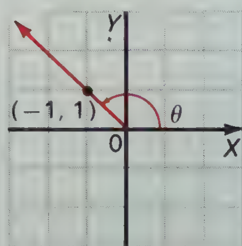


2. Make a diagram, calculate the value of r, and state the three primary ratios of the angle θ in standard position if the terminal arm passes through:

(a) $(5,-12)$ (b) $(8,-15)$ (c) $(-6,8)$
(d) $(-2,-7)$ (e) $(\sqrt{3},-1)$ (f) $(-3,-3)$

3. If θ is a second quadrant angle and $\cos\theta = -\frac{4}{5}$, state the coordinates of a point on the terminal arm and find $\sin\theta$.

4. If θ is a third quadrant angle and $\tan\theta = 1$, state the coordinates of a point on the terminal arm and find $\cos\theta$.

5. If $\sin\theta = \frac{5}{13}$, where θ is an angle in standard position:
(a) calculate the two possible values of x.
(b) make a diagram for each possibility.
(c) state the other primary trigonometric ratios.

6. If θ is a first quadrant angle and $\tan\theta = \dfrac{b}{a}$, find:
(a) $\sin\theta$ and $\cos\theta$
(b) $\sin^2\theta + \cos^2\theta$, where $\sin^2\theta$ means $(\sin\theta)^2$
(c) $\dfrac{\sin\theta}{\cos\theta}$

7. $\sin\theta = \dfrac{b}{c}$ and $\tan\theta = \dfrac{b}{a}$; make a diagram.
(a) Express c in terms of a and b.
(b) Find $\cos\theta$.
(c) Evaluate $\sin^2\theta + \cos^2\theta$.
(d) What conclusion can be made regarding $\sin^2 A + \cos^2 A$ for any angle A?
(e) Evaluate $\dfrac{\sin\theta}{\cos\theta}$
(f) What conclusion can be made regarding $\dfrac{\sin A}{\cos A}$ and $\tan A$ for any angle A?

8. The point (3,4) lies on the terminal arm of angle θ in standard position.
(a) State the coordinates of three other points on the terminal arm.
(b) Calculate the distance of each of the four points from the origin.
(c) Complete the following table in your workbook.

(x, y)	$A(3, 4)$	$B(6, ▨)$	$C(▨, 12)$	$D(▨, ▨)$
r	$\sqrt{3^2 + 4^2} = 5$			
$\sin\theta$				
$\cos\theta$				
$\tan\theta$				

Note that the values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ do not depend upon the point used on the terminal arm.

8.3 COSECANT, SECANT, AND COTANGENT

In section 8.2 we defined the **primary** trigonometric functions

$$\boxed{\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad \tan\theta = \frac{y}{x}, \quad x \neq 0}$$

We can also establish the ratios $\frac{r}{y}, \frac{r}{x}, \frac{x}{y}$. Since these ratios are the **reciprocals** of those previously studied, we now define the reciprocal trigonometric functions—cosecant, secant, cotangent:

cosecant of θ = $\csc\theta = \frac{r}{y}$, $y \neq 0$
secant of θ = $\sec\theta = \frac{r}{x}$, $x \neq 0$
cotangent of θ = $\cot\theta = \frac{x}{y}$, $y \neq 0$

EXAMPLE 1. *If $(-12, 5)$ is a point on the terminal arm of $\angle\theta$ in standard position, find the values of the six trigonometric functions.*

Solution

$$r = \sqrt{(-12)^2 + 5^2}$$
$$= \sqrt{169}$$
$$= 13$$

$$\sin\theta = \tfrac{5}{13}, \ \cos\theta = -\tfrac{12}{13}, \ \tan\theta = -\tfrac{5}{12}$$

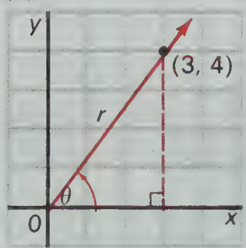
and $\csc\theta = \frac{13}{5}$, $\sec\theta = -\frac{13}{12}$, $\cot\theta = -\frac{12}{5}$

EXERCISE 8-3

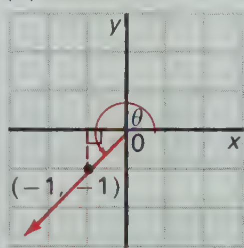
A

1. For each of the following diagrams,
 (i) State the value of r.
 (ii) State the three reciprocal ratios.

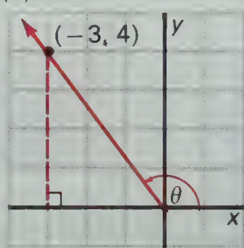
(a)



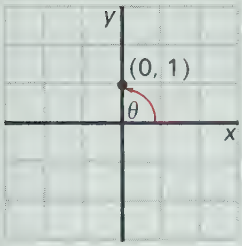
(b)



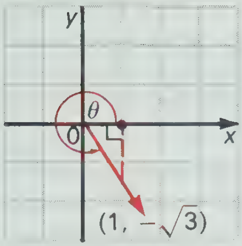
(c)



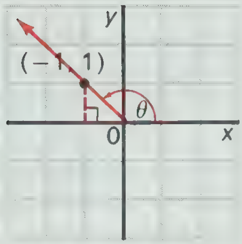
(d)



(e)



(f)



2. Make a diagram and calculate the value of r, $\csc\theta$, $\sec\theta$, and $\cot\theta$, given that each of the following is a point on the terminal arm of an angle in standard position.

(a) $(-5, -12)$ (b) $(8, 15)$ (c) $(6, -8)$
(d) $(4, -5)$ (e) $(-2, 1)$ (f) $(2, 7)$

3. If θ is a second quadrant angle and $\sec\theta = -\frac{5}{4}$, state the coordinates of a point on the terminal arm and find $\csc\theta$.

$x^2 + y^2 = r^2$

$r = \pm\sqrt{x^2 + y^2}$

4. If θ is a third quadrant angle and $\cot\theta = 1$, state the coordinates of a point on the terminal arm and find $\csc\theta$ and $\sec\theta$.

5. If $\csc\theta = \frac{13}{12}$, where θ is an angle in standard position,
(a) calculate the two possible values of x
(b) make a diagram for each possibility
(c) state $\sec\theta$ and $\cot\theta$.

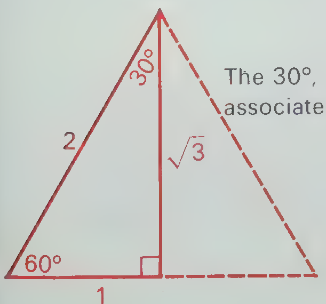
6. Make a diagram, calculate the value of r, and state the six trigonometric ratios, ($\sin\theta$, $\cos\theta$, $\tan\theta$, $\csc\theta$, $\sec\theta$, $\cot\theta$) of the angle θ in standard position if the terminal arm passes through:

(a) $(12, 5)$ (b) $(-8, 15)$ (c) $(-15, -8)$
(d) $(4, -3)$ (e) $(-6, 8)$ (f) $(5, -12)$

7. If θ is a first quadrant angle and $\cot\theta = \dfrac{a}{b}$ find
(a) $\csc\theta$ and $\sec\theta$
(b) $1 + \cot^2\theta$, and $\csc^2\theta$, where $\csc^2\theta$ means $(\csc\theta)^2$
(c) $\dfrac{\sec\theta}{\csc\theta}$

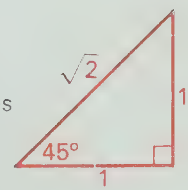
8.4 SPECIAL CASES

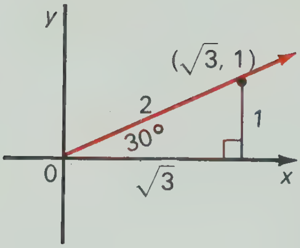
In this section we will find the trigonometric ratios of angles associated with the 30°, 60°, 90° triangle and the 45°, 45°, 90° triangle.



The 30°, 60°, 90° triangle for angles associated with 30° and 60°

The 45°, 45°, 90° triangle for angles associated with 45°





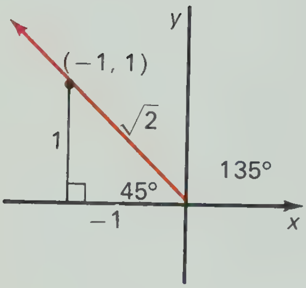
EXAMPLE 1. *Find the six trigonometric ratios of 30°.*

Solution We relate the 30°–60°–90° triangle to a pair of coordinate axes as in the diagram. Then $(\sqrt{3}, 1)$ is a point on the terminal arm and $r = 2$. The trigonometric ratios are

$$\sin\theta = \tfrac{1}{2} \qquad \csc\theta = \tfrac{2}{1}$$

$$\cos\theta = \frac{\sqrt{3}}{2} \qquad \sec\theta = \frac{2}{\sqrt{3}}$$

$$\tan\theta = \frac{1}{\sqrt{3}} \qquad \cot\theta = \frac{\sqrt{3}}{1}$$



EXAMPLE 2. *Using the 45°–45°–90° triangle find (a) sin 135°, (b) cos 135° (c) cot 135°.*

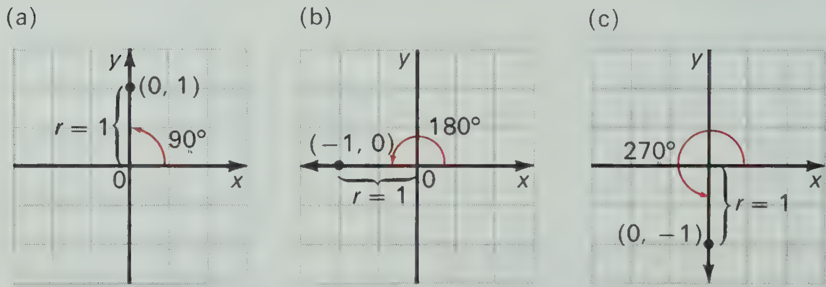
Solution We relate the 45°–45°–90° triangle to a pair of coordinate axes as in the diagram. Then $(-1, 1)$ is a point on the terminal arm and $r = \sqrt{2}$. The required trigonometric ratios are

(a) $\sin 135° = \dfrac{1}{\sqrt{2}}$ (b) $\cos 135° = \dfrac{-1}{\sqrt{2}}$ (c) $\cot 135° = \dfrac{-1}{1}$

EXAMPLE 3. *Make diagrams to find*

(a) sin 90° *(b) sec 180°* *(c) cot 270°*
tan 90° *cot 180°* *cos 270°*

Solution



$$\sin 90° = \frac{1}{1} = 1 \qquad \sec 180° = \frac{1}{-1} = -1 \qquad \cot 270° = \frac{0}{-1} = 0$$

$$\tan 90° = \frac{1}{0} \qquad \cot 180° = \frac{-1}{0} \qquad \cos 270° = \frac{0}{1} = 0$$

which is undefined which is undefined

EXERCISE 8-4

1. Find the six trigonometric ratios using a 30°–60°–90° triangle for:
(a) 60° (b) 120° (c) 150° (d) 210°

2. Use a 45°–45°–90° triangle to find the six trigonometric ratios for:
(a) 45° (b) 135° (c) 225° (d) 315°

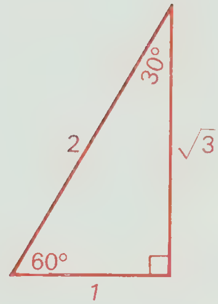
3. Use diagrams as in example 3 and state the six trigonometric ratios for:
(a) 90° (b) 180° (c) 270° (d) 360°

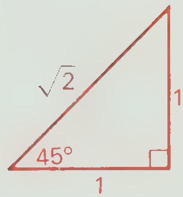
4. Evaluate the following products.
(a) sin 30° sin 60° sin 45° (b) sin 30° cos 45° tan 60°
(c) tan 30° tan 45° tan 60° (d) $\sin^2 30° \sin^2 60°$

5. Show that $\sin^2 \theta + \cos^2 \theta = 1$ for:
(a) $\theta = 30°$ (b) $\theta = 45°$ (c) $\theta = 60°$

6. (a) Evaluate
(i) sin 30° + sin 60° (ii) sin 90°
(b) Is sin 90° equal to sin 30° + sin 60°?

7. Evaluate left and right sides separately to show:
(a) sin 60° = 2 sin 30° cos 30°
(b) $\cos 60° = \sin^2 30° - \cos^2 30°$
(c) cos 90° = sin 30° sin 60° − cos 30° cos 90°





8.5 THE GRAPHS OF SIN θ, COS θ, AND TAN θ

A circle with radius 1 unit has its centre at the origin of a pair of coordinate axes.

EXERCISE 8-5

1. (a) Using points on the circumference of the unit circle in Figure 8-6, at 15° intervals, find the corresponding values of x and y to two decimal places.
(b) Complete the following table in your notebook.

	0°	15°	30°	45°	60°	75°	. . . 360°
x	1	0.97	0.87				
y	0	0.26					
r	1	1					
sin θ							
cos θ							
tan θ							

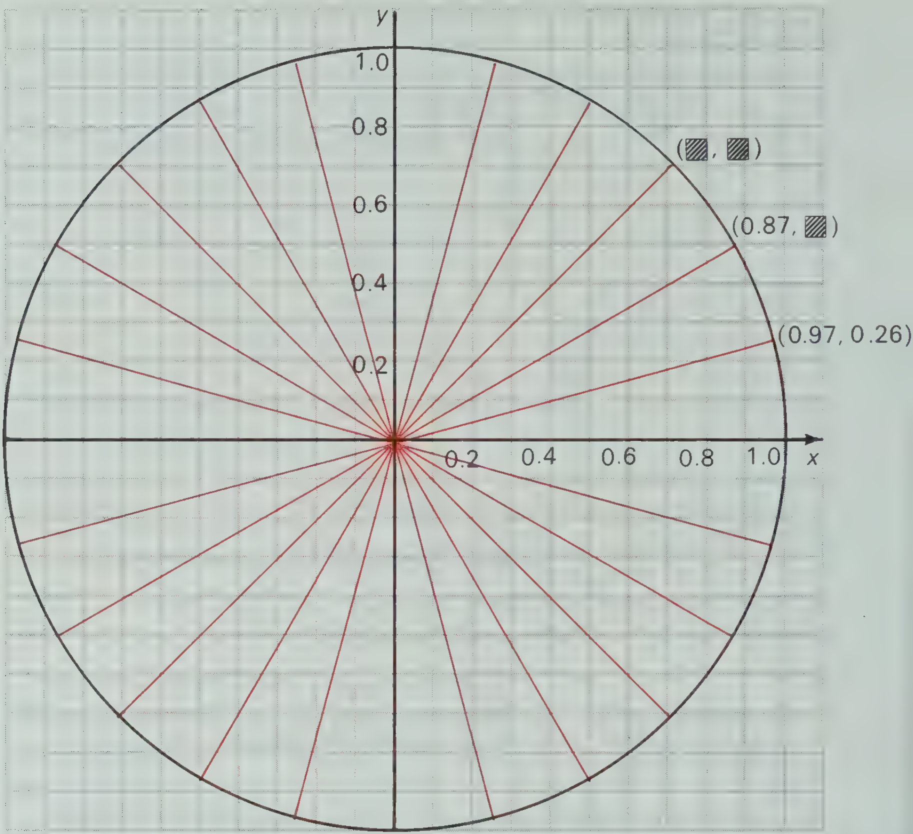


Figure 8-6

2. The graph of sin θ

(a) Using θ and sin θ axes, plot the values of sin θ from the table in question 1 and draw the curve of best fit.

(b) What is the maximum value of sin θ?

(c) What is the minimum value of cos θ?

(d) Use your graph to complete the following table.

θ			50°				215°
sin θ	0.0	0.5		0.41	−0.5	−0.87	

Since every value of θ determines a unique value for sin θ, the graph of the ordered pairs (θ, sin θ) is the graph of a function—the SINE FUNCTION.

(e) What happens to the curve for values of θ less than 0° or greater than 360°?

3. The graph of cos θ

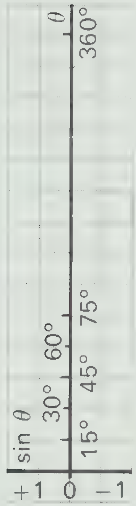
(a) Using θ and cos θ axes, plot the values of cos θ from the table in question 1 and draw the curve of best fit.

(b) What is the maximum value of cos θ?
(c) What is the minimum value of cos θ?
(d) Use your graph to complete the following table.

θ			50°				215°
cos θ	0.0	0.5		0.41	−0.5	−0.87	

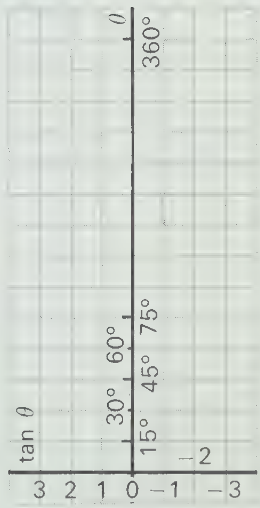
(e) What happens to the curve for values of θ less than 0° or greater than 360°?

Since every value of θ determines a unique value for cos θ, the graph of the .ordered pairs (θ, cos θ) is the graph of a function—the COSINE FUNCTION.



4. The graph of tan θ
(a) On θ and tan θ axes, draw dotted lines parallel to the tan θ-axis at the values of θ for which tan θ is undefined. Since tan θ is not defined for these values of θ, no point on the graph of tan θ can lie on these dotted lines.
(b) Plot the values of tan θ from the table in question 1 and draw the curve of best fit.
(c) Use your graph to complete the following table.

θ						135°	330°
tan θ	0	1.0	−1.0	0.7	1.7		



(d) What happens to the curve for values of θ less than 0° or greater than 360°?

Since every value of θ for which tan θ is defined determines a unique value for tan θ, the graph of the ordered pairs (θ, tan θ) is the graph of a function—the TANGENT FUNCTION.

5. (a) On the same set of axes $0° \leq \theta \leq 360°$, draw the graphs of
(i) sin θ (ii) cos θ
(b) State the values of θ for which
$$\sin \theta = \cos \theta$$

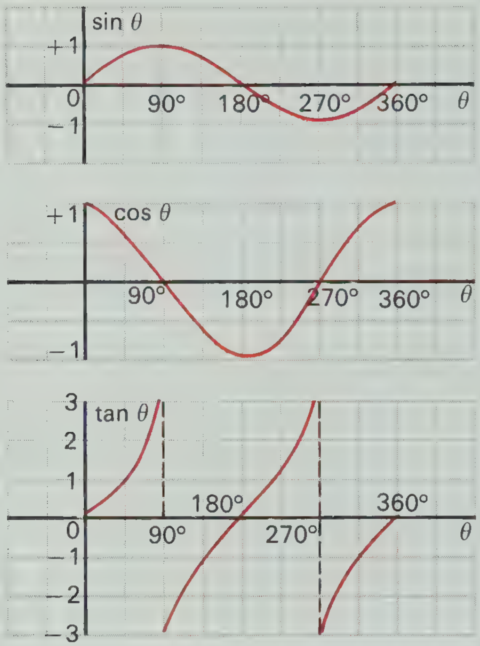
6. Draw a graph for $-\sin \theta$ taking values of θ from 0° to 360° at 30° intervals.

7. (a) On the same axes, $0° \leq \theta \leq 360°$, draw the graphs of
(i) sin θ (ii) cos θ (iii) sin θ + cos θ
(b) State the value(s) for which sin θ + cos θ is
(i) a maximum (ii) a minimum (iii) zero

The sine, cosine, and tangent functions are called the primary trigonometric functions.

8.6 THE SIGN OF SIN θ, COS θ, AND TAN θ

It is convenient to be able to tell whether a trigonometric ratio will be positive or negative by looking at the angle.



If θ lies in the first quadrant (0° to 90°) or in the second quadrant (90° to 180°), sin θ is positive for these values.

If θ lies in the first quadrant (0° to 90°) or in the fourth quadrant (270° to 360°), cos θ is positive for these values.

If θ lies in the first quadrant (0° to 90°) or in the third quadrant (180° to 270°), tan θ is positive for these values.

These results can be summarized in the following diagram

Sine positive	All positive
Tangent positive	Cosine positive

This memory aid is called the CAST RULE.

EXERCISE 8-6

A 1. If sin θ and tan θ are both positive, in what quadrant does θ lie?

2. If cos θ and tan θ are both negative, what is the sign of sin θ?

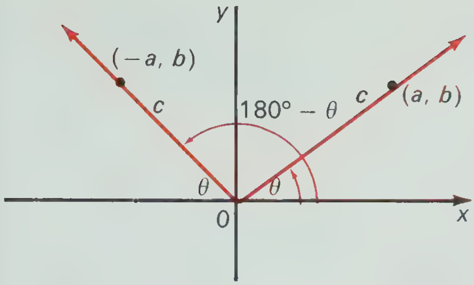
3. State the sign of the following:

(a) sin 225°	(b) cos 318°	(c) tan 272°	(d) sin 135°
(e) tan 305°	(f) tan 204°	(g) sin 310°	(h) cos 170°
(i) sin 85°	(j) tan 175°		

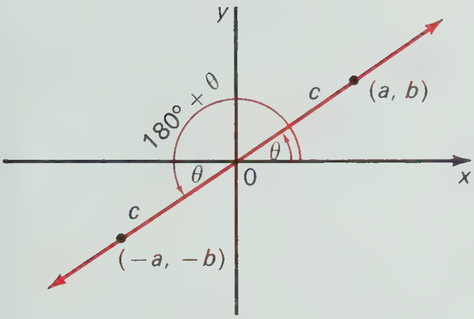
8.7 USE OF TRIGONOMETRIC TABLES

Tables in the book are provided for angles from 0° to 90°. The trigonometric ratios of angles greater than 90° are related to the ratios of acute angles as follows.

Test your skill:
$1.001 \times 0.005 \div 5$



$$\frac{b}{c} = \sin\theta \qquad \frac{a}{c} = \cos\theta \qquad \frac{b}{a} = \tan\theta$$



$$\sin(180° - \theta) = \frac{b}{c} = +\sin\theta$$

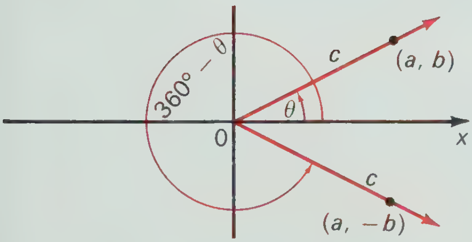
$$\cos(180° - \theta) = -\frac{a}{c} = -\cos\theta$$

$$\tan(180° - \theta) = \frac{b}{-a} = -\tan\theta$$

$$\sin(180° + \theta) = -\frac{b}{c} = -\sin\theta$$

$$\cos(180° + \theta) = -\frac{a}{c} = -\cos\theta$$

$$\tan(180° + \theta) = \frac{-b}{-a} = +\tan\theta$$



$$\sin(360° - \theta) = \frac{-b}{c} = -\sin\theta$$

$$\cos(360° - \theta) = \frac{a}{c} = +\cos\theta$$

$$\tan(360° - \theta) = \frac{-b}{a} = -\tan\theta$$

These diagrams are also used to derive the following relationships:

$\csc(180° - \theta) = \csc\theta$ $\qquad$ $\csc(180° + \theta) = -\csc\theta$
$\sec(180° - \theta) = -\sec\theta$ $\qquad$ $\sec(180° + \theta) = -\sec\theta$
$\cot(180° - \theta) = -\cot\theta$ $\qquad$ $\cot(180° + \theta) = +\cot\theta$

$\csc(360° - \theta) = -\csc\theta$
$\sec(360° - \theta) = +\sec\theta$
$\cot(360° - \theta) = -\cot\theta$

EXAMPLE 1. *Find (a) sin 140° (b) sec 215° (c) cot 330°*

Solution

(a) $\sin 140° = \sin(180° - 40°)$ $\qquad$ (b) $\sec 215° = \sec(180° + 35°)$

$= \sin 40°$

$\doteq 0.6428$ (from tables)

$= -\sec 35°$

$\doteq -1.2208$

(c) $\cot 330° = \cot (360° - 30°)$

$= -\cot 30°$

$\doteq -0.5774$

EXERCISE 8-7

A 1. Insert + or − signs in the blanks to make the following statements true.

(a) $\sin 220° =$ ▨$\sin 40°$ (b) $\cos 137° =$ ▨$\cos 43°$
(c) $\cos 301° =$ ▨$\cos 59°$ (d) $\tan 260° =$ ▨$\tan 80°$
(e) $\sin 100° =$ ▨$\sin 80°$ (f) $\tan 350° =$ ▨$\tan 10°$
(g) $\cos 200° =$ ▨$\cos 20°$ (h) $\sin 270° =$ ▨$\sin 90°$
(i) $\cos 180° =$ ▨$\cos 0°$ (j) $\tan 225° =$ ▨$\tan 45°$

2. Insert + or − signs in the blanks to make the following statements true.

(a) $\csc 140° =$ ▨$\csc 40°$ (b) $\cot 325° =$ ▨$\cot 35°$
(c) $\sec 220° =$ ▨$\sec 40°$ (d) $\csc 315° =$ ▨$\csc 45°$
(e) $\cot 240° =$ ▨$\cot 60°$ (f) $\sec 330° =$ ▨$\sec 30°$

3. From the tables, find:

(a) $\sin 24°$ (b) $\cos 71°$ (c) $\tan 28°$ (d) $\cos 32°$
(e) $\sin 58°$ (f) $\sin 32°$ (g) $\cos 40°$ (h) $\cos 50°$
(i) $\sin 90°$ (j) $\tan 31°$ (k) $\cos 83°$ (l) $\tan 90°$

4. From the tables, find:

(a) $\csc 58°$ (b) $\sec 75°$ (c) $\cot 53°$ (d) $\sec 32°$
(e) $\csc 29°$ (f) $\csc 18°$ (g) $\sec 50°$ (h) $\sec 40°$
(i) $\sec 90°$ (j) $\cot 69°$ (k) $\sec 73°$ (l) $\cot 90°$

S	A
T	C

B 5. Using the tables, find:

(a) $\sin 145°$ (b) $\cos 138°$ (c) $\tan 125°$ (d) $\sin 230°$
(e) $\sin 205°$ (f) $\tan 235°$ (g) $\sin 325°$ (h) $\cos 208°$
(i) $\cos 222°$ (j) $\tan 305°$ (k) $\cos 318°$ (l) $\tan 215°$
(m) $\sin 140°$ (n) $\tan 155°$ (o) $\cos 160°$ (p) $\tan 245°$

6. Using the tables, find:

(a) $\csc 300°$ (b) $\sec 315°$ (c) $\cot 335°$ (d) $\csc 320°$
(e) $\csc 100°$ (f) $\cot 115°$ (g) $\sec 135°$ (h) $\sec 340°$
(i) $\csc 120°$ (j) $\cot 295°$ (k) $\csc 110°$ (l) $\cot 120°$

EXAMPLE 2. *Find two values for A, given $\cos A = -0.8746$.*

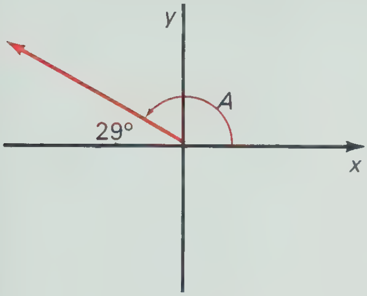
Solution

$\angle A$ lies in the second or third quadrant.

From tables,

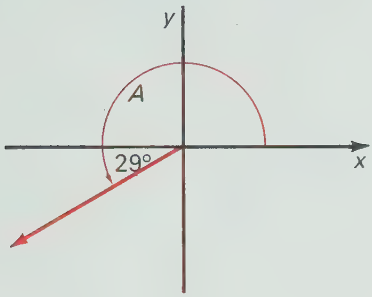
$0.8746 = \cos 29°$

(i) Second Quadrant



$\angle A = 180° - 29°$
$= 151°$

(ii) Third Quadrant



$\angle A = 180° + 29°$
$= 209°$

7. From the tables, find two values for each of the indicated angles.

(a) $\cos A \doteq 0.7430$ (b) $\sin B \doteq 0.8910$ (c) $\cos C \doteq -0.4540$
(d) $\tan D \doteq 0.9000$ (e) $\sin E \doteq 0.6020$ (f) $\tan F \doteq -1.6000$
(g) $\cos G \doteq -0.9703$ (h) $\sin H \doteq -0.9703$ (i) $\tan J \doteq 4.7050$

8.8 TRIGONOMETRIC RATIOS OF THE ACUTE ANGLES OF A RIGHT TRIANGLE

In Figure 8-7, θ is an angle of the right triangle AOB and the triangle is located so that $\angle\theta$ is in standard position. With reference to θ:

BA is the side opposite to θ,
OB is the side adjacent to θ,
OA is the hypotenuse of the triangle.

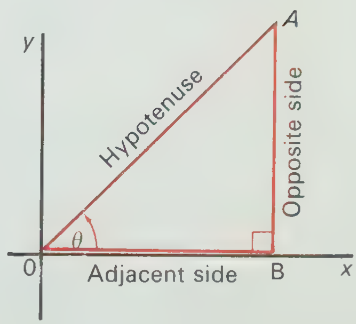


Figure 8-7

Thus, for θ in the right triangle AOB:

Primary Trigonometric Ratios

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

Reciprocal Trigonometric Ratios

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite side}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent side}}$$

$$\cot\theta = \frac{\text{adjacent side}}{\text{opposite side}}$$

Note: θ must be an acute angle of a right triangle.

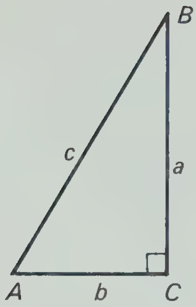


Figure 8-8

In Figure 8-8, $\triangle ABC$ is a right triangle with $\angle C = 90°$. Notice that the side CB whose length is a is opposite the vertex A. Similarly b is opposite the angle B. A can represent the angle at vertex A if the meaning is clear. Thus, for the triangle ABC

$$\sin A = \frac{a}{c}, \qquad \cos A = \frac{b}{c}, \qquad \tan A = \frac{a}{b}$$

$$\sin B = \frac{b}{c}, \qquad \cos B = \frac{a}{c}, \qquad \tan B = \frac{b}{a}$$

$$\csc A = \frac{c}{a}, \qquad \sec A = \frac{c}{b}, \qquad \cot A = \frac{b}{a}$$

$$\csc B = \frac{c}{b}, \qquad \sec B = \frac{c}{a}, \qquad \cot B = \frac{a}{b}$$

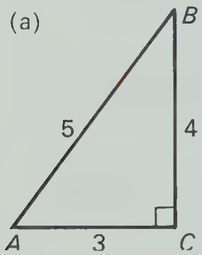
where $c = \sqrt{a^2 + b^2}$

EXERCISE 8-8

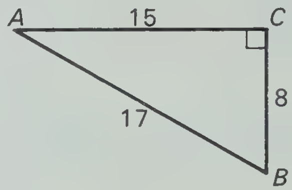
A

1. State the primary trigonometric ratios for $\angle A$ in each of the following triangles:

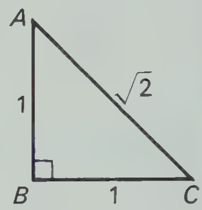
(a)



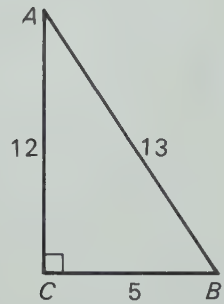
(b)



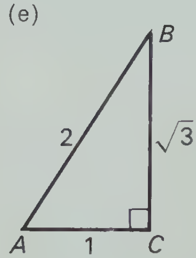
(c)



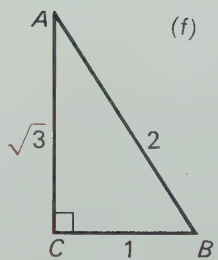
(d)



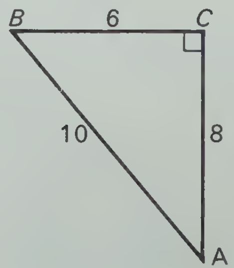
(e)



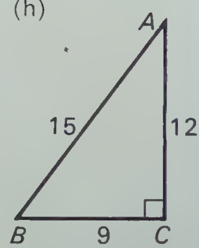
(f)



(g)



(h)



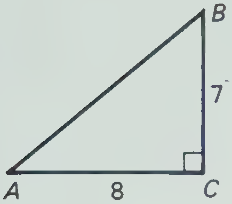
Starting with the word "swim" and changing one letter at a time to form a new word, can you reach "flat" in four changes?

swim
1. —
2. —
3. —
4. flat

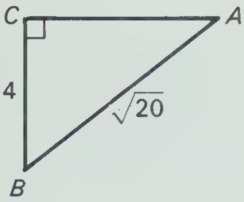
2. State the reciprocal ratios for $\angle B$ in each triangle of question 1.

3. State a primary trigonometric ratio from which $\angle A$ can be determined for each of the following triangles.

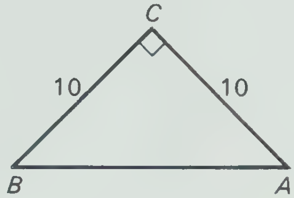
(a)



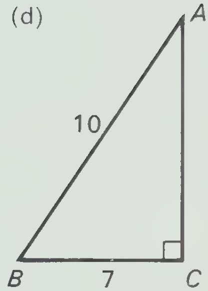
(b)



(c)

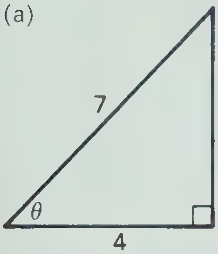


(d)

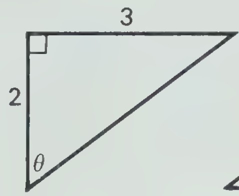


4. State a reciprocal ratio from which the indicated angle can be determined for each of the following triangles.

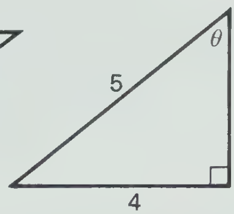
(a)



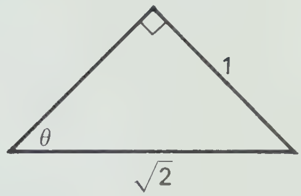
(b)



(c)



(d)



B

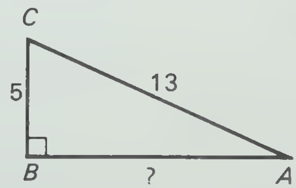
5. Without simplifying find (i) the primary trigonometric ratios, (ii) the reciprocal trigonometric ratios for the acute angles of the following triangles.

(a) $\triangle ABC$, $\angle B = 90°$, $AB = 3$ cm, $BC = 4$ cm

(b) $\triangle DEF$, $\angle E = 90°$, $DE = 4$ cm, $DF = \sqrt{17}$ cm

(c) $\triangle JKL$, $\angle J = 90°$, $JK = 5$ cm, $KL = 13$ cm

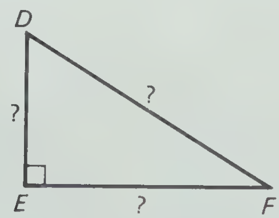
(d) $\triangle PQR$, $PQ = 7$ cm, $QR = 24$ cm, $PR = 25$ cm



6. In $\triangle ABC$, $\sin A = \frac{5}{13}$ and $\angle B = 90°$.

(a) Find the primary trigonometric ratios of $\angle A$ and $\angle C$.

(b) Find the reciprocal trigonometric ratios of $\angle A$ and $\angle C$.

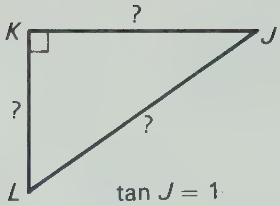


tan $F = 0.75$

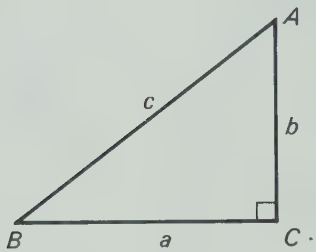
7. In $\triangle DEF$, $\tan F = 0.75$ and $\angle E = 90°$.

(a) Find the primary trigonometric ratios of $\angle D$ and $\angle F$.

(b) Find the reciprocal trigonometric ratios of $\angle D$ and $\angle F$.



8. In $\triangle JKL$, $\tan J = 1$ and $\angle K = 90°$.
(a) Find the primary trigonometric ratios of $\angle J$ and $\angle L$.
(b) Find the reciprocal trigonometric ratios of $\angle J$ and $\angle L$.

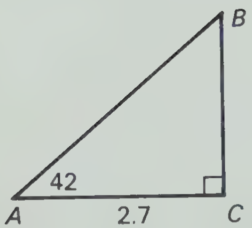


$a^2 + b^2 = c^2$

9. (a) State the primary ratios of the acute angles in $\triangle ABC$.
(b) Find the value of $\sin^2 \theta + \cos^2 \theta$.
(c) Find $\dfrac{\sin A}{\cos A}$ and express this ratio in terms of $\tan A$.
(d) Make a new diagram with $c = 1$.
  (i) What function is a of $\angle A$?
  (ii) What function is b of $\angle B$?

8.9 SOLVING RIGHT TRIANGLES I

A triangle is solved when you can state:
 (i) the lengths of the three sides; and
 (ii) the measure of the three angles.



EXAMPLE 1. *In $\triangle ABC$, $\angle C = 90°$, $b = 2.7$ units and $\angle A = 42°$. Solve the triangle by finding $\angle B$, a and c.*

Solution (i) $\angle B = 90° - 42° = 48°$

(ii) $\dfrac{a}{2.7} = \tan 42°$

$a \doteq 2.7 \times 0.9004$ (from tables)

$a \doteq 2.43$

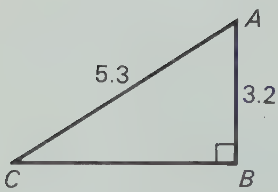
(iii) $c^2 = a^2 + b^2$

$= (2.43)^2 + (2.7)^2$

$\doteq 5.90 + 7.29$

$\doteq \sqrt{13.19}$

$c \doteq 3.62$



EXAMPLE 2. *In $\triangle ABC$, $\angle B = 90°$, $AB = 3.2$ units, and $AC = 5.3$ units. Solve the triangle.*

Solution (i) $\sin C = \dfrac{3.2}{5.3} \doteq 0.604$ (by slide rule)

$\sin 37° \doteq 0.6018$ (from tables)

$\sin 38° \doteq 0.6157$ (from tables)

$\therefore \angle C \doteq 37°$ (to the nearest degree)

$\angle A \doteq 90° - 37° = 53°$

(ii) $BC^2 = (5.3)^2 - (3.2)^2$

$\doteq 28.1 - 10.2 \doteq 17.9$

$BC \doteq 4.23$

Alternate method:

$$\frac{BC}{5.3} = \cos 37°$$

$$BC = 5.3 \times \cos 37°$$

$$\doteq 5.3 \times 0.7986 \doteq 4.23$$

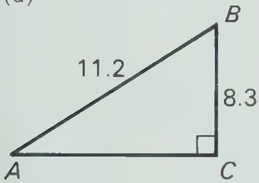
How many diagonals can you draw in an OCTAGON?

EXERCISE 8-9

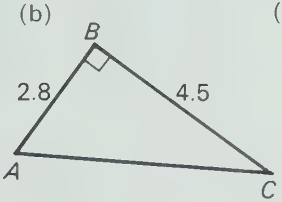
B

1. Find $\angle A$ in each of the following by using a trigonometric ratio.

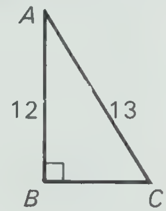
(a)



(b)

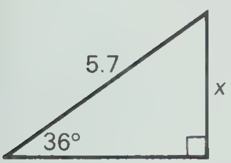


(c)

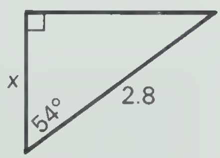


2. Find x in each of the following.

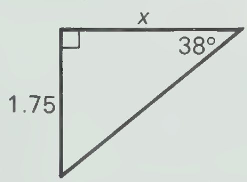
(a)



(b)



(c)

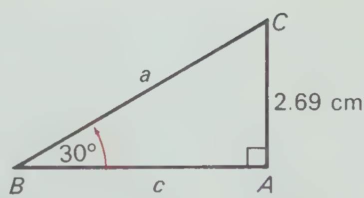


3. Solve the following triangles. Give all angle measurements to the nearest degree and all lengths to two-figure accuracy.

(a) $\angle A = 90°$, $b = 3.0$, $c = 4.0$
(b) $\angle C = 90°$, $c = 1.7$, $b = 1.5$
(c) $\angle B = 90°$, $a = 36$, $c = 21$
(d) $\angle B = 90°$, $\angle C = 36°$, $a = 1.9$
(e) $\angle A = 90°$, $\angle B = 40°$, $a = 9.4$
(f) $\angle C = 90°$, $a = 2.4$, $b = 3.7$
(g) $\angle A = 90°$, $\angle C = 71°$, $a = 1.0$
(h) $\angle B = 90°$, $a = 2.6$, $c = 5.8$

8.10 SOLVING RIGHT TRIANGLES II

The examples and problems of section 8.9 were limited to cases where only the sine, cosine, and tangent ratios were required. In this section we can use the reciprocal ratios (cosecant, secant, and cotangent).



EXAMPLE 1. *Solve $\triangle ABC$, $\angle A = 90°$, $\angle B = 30°$, $AC = 2.69$ cm.*

Solution

$$\angle C = 180° - (90° + 30°)$$
$$= 60°$$

$$\frac{a}{2.69} = \csc 30°$$

$$a \doteq 2.69 \times 2.00 \qquad \text{(from tables)}$$
$$\doteq 5.38$$

$$\frac{c}{2.69} = \cot 30°$$

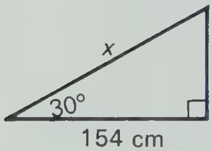
$$c \doteq 2.69 \times 1.732 \qquad \text{(from tables)}$$
$$\doteq 4.66$$

$a \doteq 5.38$ cm, $c = 4.66$ cm, and $\angle C = 60°$

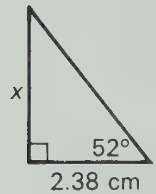
EXERCISE 8-10

B 1. Find the length of the side labelled x in each of the following.

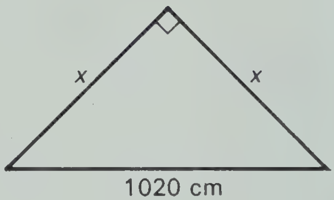
(a)



(b)



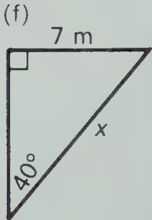
(c)



(d)

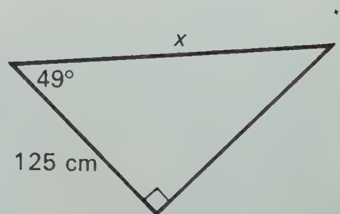
(e)

(f)



(g)

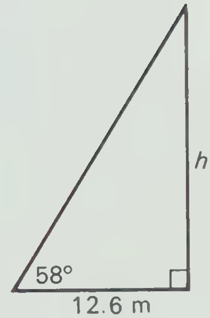
(h)



2. Solve the following triangles. Give all angle measurements to the nearest degree and all lengths to two-figure accuracy.
(a) $\triangle ABC$, $\angle A = 90°$, $\angle C = 35°$, $AC = 120$ cm
(b) $\triangle DEF$, $\angle E = 90°$, $DE = 32.5$ cm, $EF = 24.5$ cm
(c) $\triangle GHJ$, $\angle J = 90°$, $JG = 10.0$ cm, $HG = 14.4$ cm
(d) $\triangle KLM$, $\angle L = 90°$, $\angle K = 60°$, $LM = 1.73$ cm

8.11 APPLICATIONS OF TRIGONOMETRY

EXAMPLE 1. *The angle of elevation of the top of a building is 58° from a point 12.6* m *from the foot of the building. Find the height of the building.*



Solution Let the height of the building in metres be represented by h.

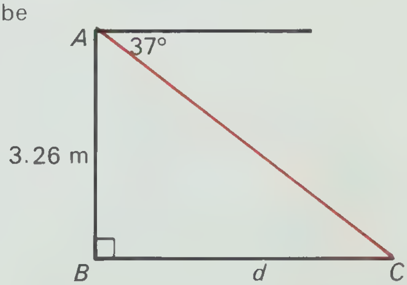
$$\frac{h}{12.6} = \tan 58°$$

$$h = 12.6 \times \tan 58°$$

$$\doteq 12.6 \times 1.6003 \doteq 20.2$$

The height of the building is approximately 20.2 m.

EXAMPLE 2. *From the top of a building 3.26* m *high, the angle of depression of a certain car is observed to be 37°. How far is the car from the building?*



Solution Let the distance from the car to the building in metres be represented by d.

$$\angle BAC = 90° - 37° = 53°$$

$$\frac{d}{3.26} = \tan 53°$$

$$d = 3.26 \times \tan 53°$$

$$\doteq 3.26 \times 1.3270 \doteq 4.33$$

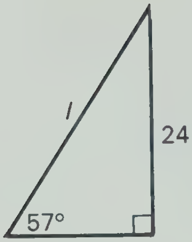
The car is approximately 4.33 m from the building.

Examples 1 and 2 suggest a sequence of steps that may be used for the successful solution of problems.

1. Make a reasonably accurate diagram and mark all given data on it.
2. Represent the unknown quantity by a variable.
3. Set up an equation using

$$\boxed{\frac{\text{unknown}}{\text{known}} = \text{trig. ratio.}}$$

4. Find the value of the variable using a slide rule or calculator.
5. Write your conclusion, stating your answer to the accuracy warranted by the question. Is the answer reasonable?



EXAMPLE 3. *A guy wire is attached to a tower at a point 24* m *above the ground and makes an angle of 57° with the ground. Calculate the length of cable required to the nearest metre, including an extra 4* m *for connections.*

Solution Let the length in metres of the guy wire be *l*.

$$\frac{l}{24} = \csc 57°$$

$$l \doteq 24 \times 1.192 \doteq 28.6$$

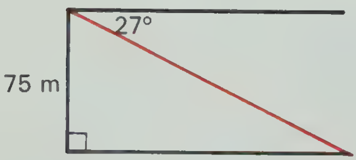
$$l + 4 \doteq 32.6$$

The length of cable required is 33 m.

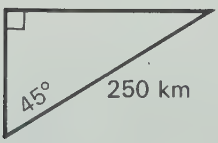
EXERCISE 8-11

B

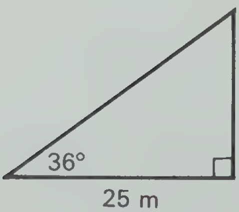
1. The angle of elevation of the top of a building is 74° from a point 65 m from the foot of the building. Find the height of the building to the nearest metre.



2. From the top of a fire tower, the angle of depression of a cabin is observed to be 27°. Find the distance from the cabin to the tower if the tower is 75 m high.



3. A ship sails 250 km in a north-easterly direction. How far north has it travelled?

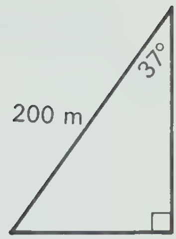


4. A tree casts a 25 m shadow when the elevation of the sun is 36°. Calculate to the nearest 0.5 m:

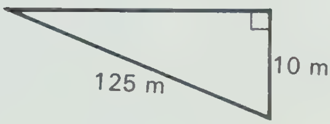
(a) the height of the tree

(b) the length of the shadow when the elevation of the sun is 58°.

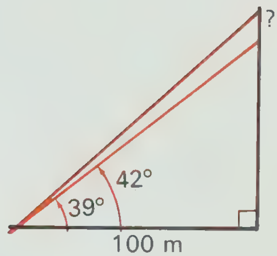
5. A 200 m guy wire makes an angle of 37° at the top of a radio tower. Calculate the height of the tower.

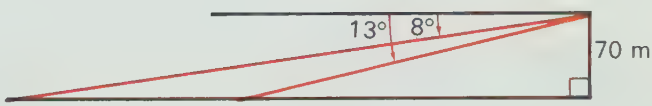


6. A road drops 10 m for every 125 m of road surface. Calculate the angle of inclination of the road.



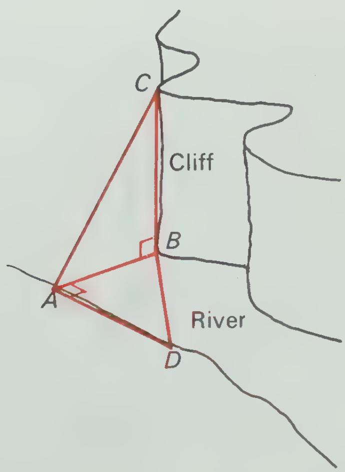
7. From a point 100 m from the foot of a building, the angles of elevation of the top and bottom of the building's flagpole are 42° and 39° respectively. Calculate the height of:
(a) the building (b) the flagpole.

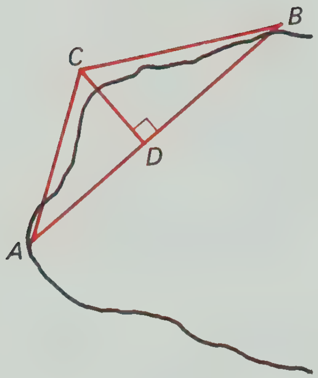




8. From the top of a cliff 70 m high the angles of depression of two small boats on the water are 8° and 13°. Calculate the distance:
(a) from the bottom of the cliff to the closer boat
(b) between the boats.

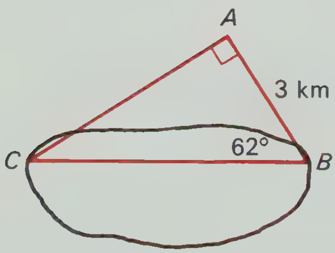
9. A surveyor wishes to find the height BC of an inaccessible cliff. To do this, he sets up his transit at A, measures $\angle CAB$, lays off a base line AD perpendicular to AB, and measures $\angle D$. From such a procedure, he records the following data: $\angle CAB = 66°$, $\angle ADB = 42°$, $AD = 34.5$ m. Calculate the height of the cliff.



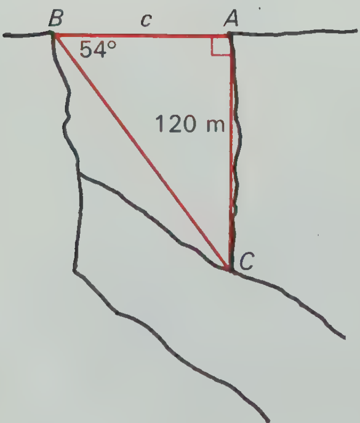


10. From two points A and B on opposite sides of a bay, the distances to a point C were measured and found to be 1300 m and 1900 m respectively. If $\angle A = 32°$ and $\angle B = 28°$, calculate the distance AB across the bay.

11. How far is it to the foot of a 400 m tower if the angle of elevation of the top is 35°?

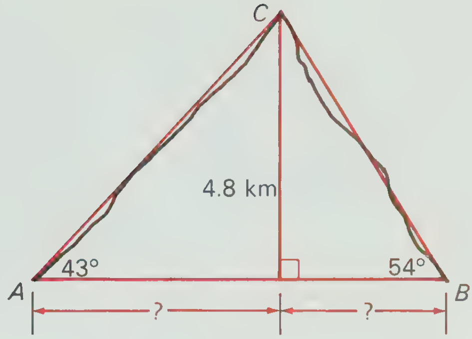


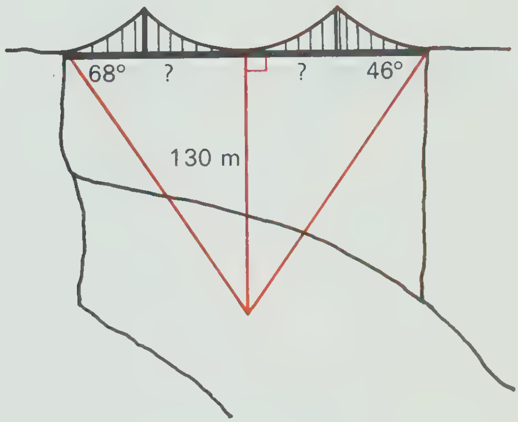
12. Find the distance across a lake, BC, if $\angle ABC = 62°$, $AB = 3$ km and $\angle BAC = 90°$.



13. Find the distance across a gorge, AB, if you are standing at B, and the angle of depression of the base of the opposite side is 54°. Both sides are 120 m high.

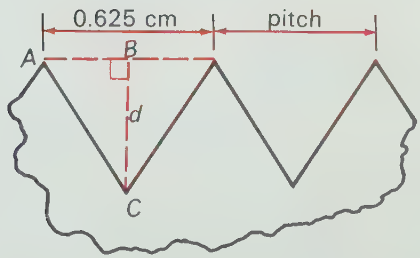
14. Mont Blanc, the highest of the Alps, is 4.8 km high. If the angles of elevation from points A and B on opposite sides are 43° and 54° as in the diagram, calculate the length of the tunnel AB.





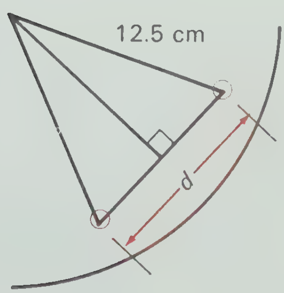
15. A bridge is 130 m above the water. From the ends of the bridge, the angles of depression of a boat in the water directly between the points are 46° and 68°. Find the width of the bridge.

Shop Problems

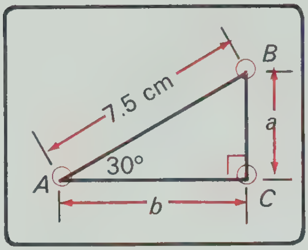


The threads form an equilateral triangle.

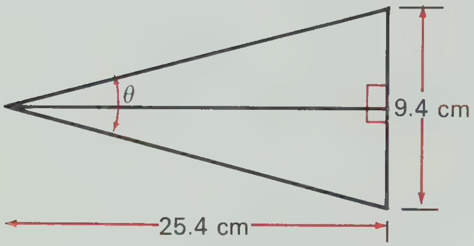
16. Calculate the depth of cut, d, of a sharp V thread if the pitch of the thread is 0.625 cm.



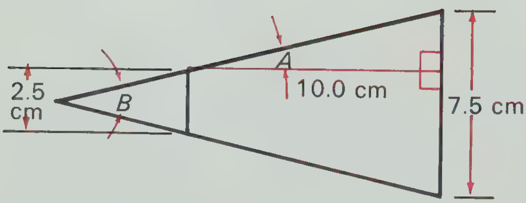
17. Calculate the distance between centres of the holes on a 25 cm bolt circle containing nine holes.



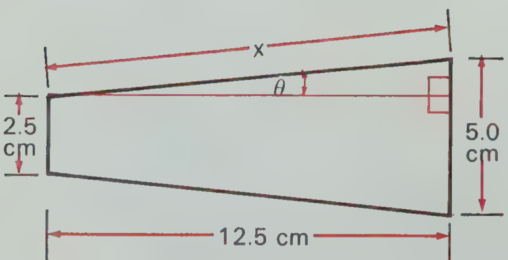
18. Three holes are to be located in a rectangular plate as shown. Find the dimensions *a* and *b*.



19. Find the indicated angle in the wedge as shown. Use a right triangle.

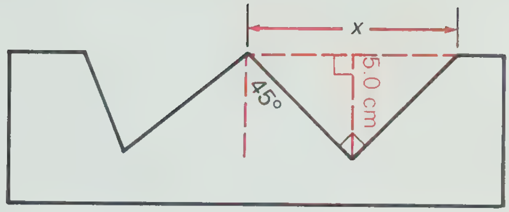


20. Calculations for tapers are similar to those for wedges. Calculate the angles marked *A* and *B* to the nearest degree for the conical taper to be turned on a lathe.

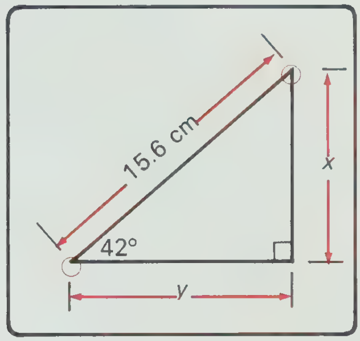


21. (a) Find the angle marked θ.
(b) Find the length marked *x*.

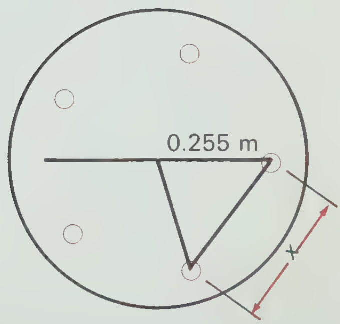
22. Calculate the measurement x in the given diagram of a template.



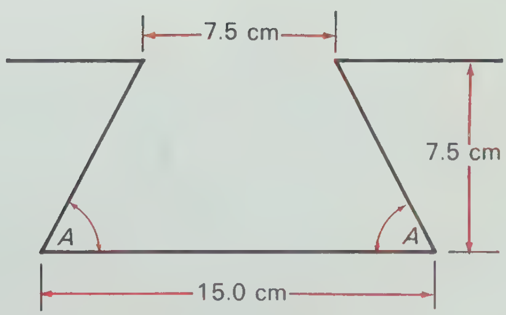
23. In the given plate, two holes must be drilled as shown. Calculate the dimensions marked x and y if the distance between centres is 15.6 cm.

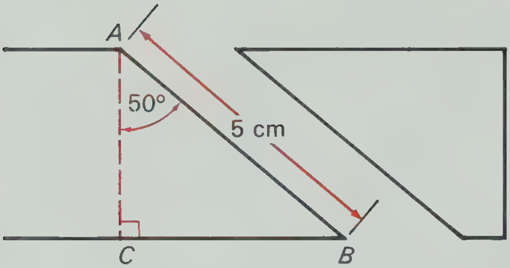


24. Calculate the distance between centres of two adjacent holes on a five-hole bolt circle with a diameter 0.255 m.

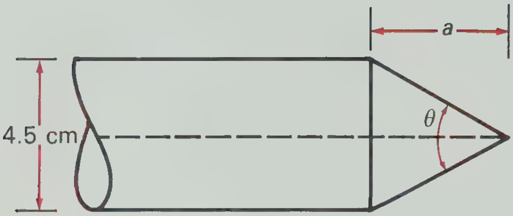


25. Find the indicated angles, A, in the dovetail slide in the diagram.

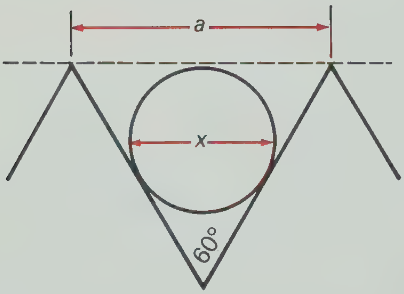




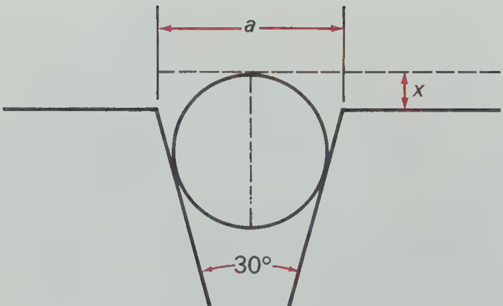
26. If the length of the hole, AB, is 5 cm, and the hole is at an angle of 50°, calculate the thickness of the material AC.



27. In the diagram of the bit shown, $a = 4.0$ cm. Calculate θ, the angle of taper, to the nearest degree.

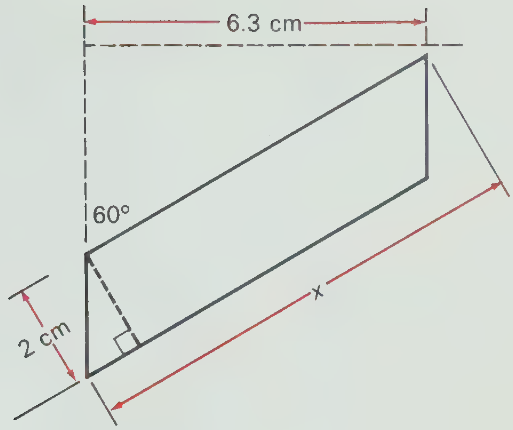


28. If $a = 6.0$ cm, calculate the diameter x.



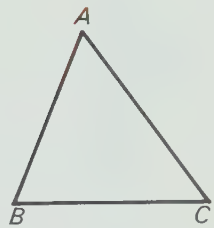
29. The diagram shows a partial view of the Acme thread (or worm). The diameter of the circle in the two-dimensional diagram is 1.5 cm. If $a = 1.75$ cm, find dimension x.

30. The diagram shows the end view of a jib. Calculate the dimension x.

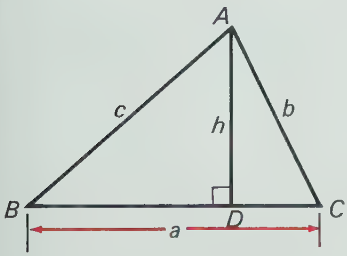


8.12 THE LAW OF SINES

Triangles that do not contain a right angle are called oblique triangles. In solving right triangles, three of the six parts (including at least one side) of the triangle were given. The remaining three parts could be found using trigonometric ratios. In this section we will solve oblique triangles using a general formula—the law of sines.



(i) Acute Triangle



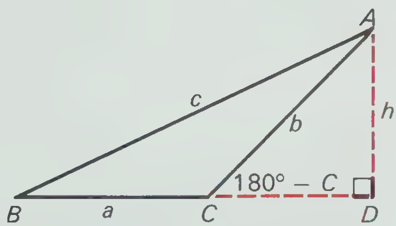
In $\triangle ACD$, $\dfrac{h}{b} = \sin C$

$h = b \sin C$

In $\triangle ABD$, $\dfrac{h}{c} = \sin B$

$h = c \sin B$

(ii) Obtuse Triangle



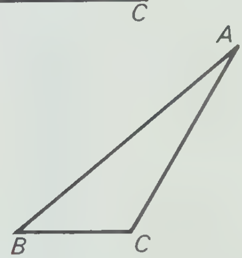
In $\triangle ACD$, $\dfrac{h}{b} = \sin(180° - C)$

$= \sin C$

$h = b \sin C$

In $\triangle ABD$, $\dfrac{h}{c} = \sin B$

$h = c \sin B$



$\sin(180° - C) = \sin C$

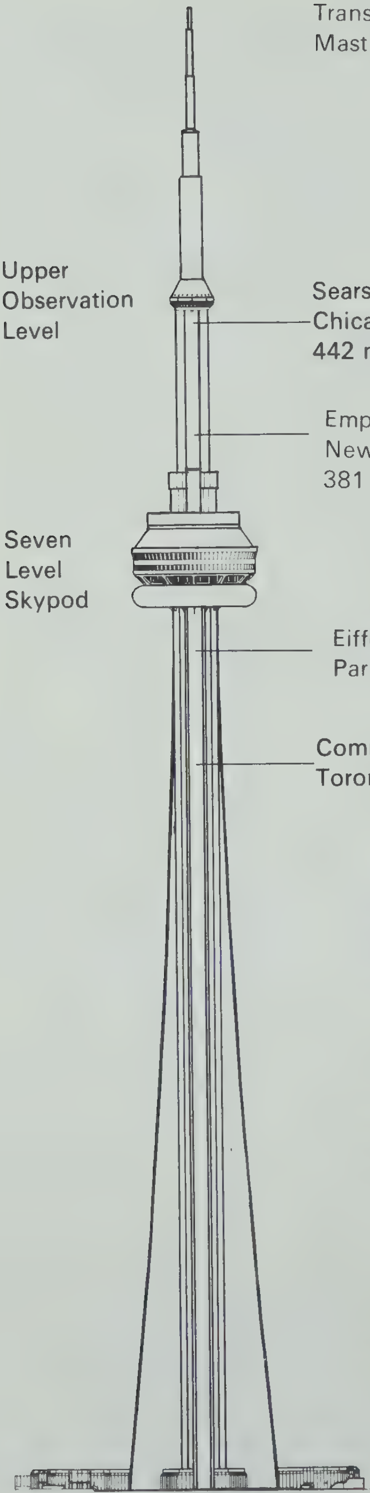
For both the acute and obtuse triangles,

$$b \sin C = c \sin B$$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

By drawing the altitude from C, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$



The CN Tower

The imposing and elegant structure near the Toronto waterfront is the superbly designed and proportioned CN tower. At 550 m, it is the world's tallest free-standing structure. Opened in 1976, the tower consists of 446 m of slip-form concrete, topped with a transmission mast over 100 m above. Other features and specifications are:

Seven-level skypod
Dining room level — — — — —348 m
Observation levels — — — — —335 m
Visibility — — — — — — — — — — —120–130 km
Upper observation level — —457 m
Capacity
Skypod dining room — — — —400 people
Observation areas — — — — —600 people
Elevators — — — — — — — — — — —1300 people per hour
Broadcast facilities — — — —8 TV Channels
11 FM radio stations
Total mass of tower — — — —118 000 t
Variance from plumb — — —2.8 cm

The figure shows the height of the CN Tower relative to other noteworthy free-standing structures in the world at the time of its construction.

The results are the same for both the acute and obtuse triangles:

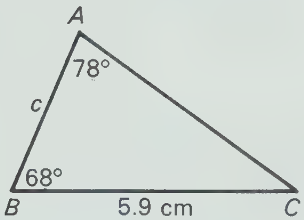
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

EXAMPLE 1. *In $\triangle ABC$, $\angle A = 78°$, $\angle B = 68°$, and $a = 5.9$ cm. Find c.*

Solution

$\sin(180° - C) = \sin C$

$$\begin{aligned} \angle C &= 180° - (78° + 68°) \\ &= 34° \end{aligned}$$



$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\begin{aligned} c &= \frac{a \sin C}{\sin A} \\ &\doteq \frac{5.9 \times 0.5592}{0.9782} \\ &\doteq 3.37 \end{aligned}$$

$\frac{p}{q} = \frac{r}{s}$

$p = \frac{rq}{s}$

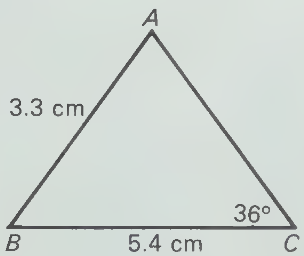
The length of c is 3.4 cm.

EXAMPLE 2. *In $\triangle ABC$, $c = 3.3$ cm, $\angle C = 36°$, and $a = 5.4$ cm. Find $\angle A$, given that $\triangle ABC$ is acute.*

Solution

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

$$\begin{aligned} \sin A &= \frac{a \sin C}{c} \\ &\doteq \frac{5.4 \times 0.5878}{3.3} \\ &\doteq 0.9619 \\ \angle A &\doteq 74° \end{aligned}$$



If example 2 had stated that the triangle was obtuse, then we would have used the $\sin(180° - \theta) = \sin \theta$ relationship, and

$$\begin{aligned} \angle A &\doteq 180° - 74° \\ &\doteq 106° \end{aligned}$$

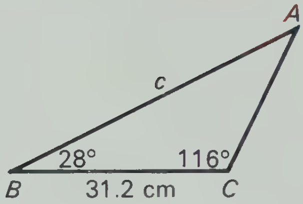
EXAMPLE 3. *In $\triangle ABC$, $\angle B = 28°$, $\angle C = 116°$, and $a = 31.2$ cm. Find c.*

θ	$\sin\theta$
0	0.0000
1	0.0175
2	0.0349
3	0.0523
4	0.0698
5	0.0872
6	0.1045
7	0.1219
8	0.1392
9	0.1564
10	0.1736
11	0.1908
12	0.2079
13	0.2250
14	0.2419
15	0.2588
16	0.2756
17	0.2924
18	0.3090
19	0.3256
20	0.3420
21	0.3584
22	0.3746
23	0.3907
24	0.4067
25	0.4226
26	0.4384
27	0.4540
28	0.4695
29	0.4848
30	0.5000
31	0.5150
32	0.5299
33	0.5446
34	0.5592
35	0.5736
36	0.5878
37	0.6018
38	0.6157

Solution

$$A = 180° - (28° + 116°)$$
$$= 36°$$
$$\frac{c}{\sin C} = \frac{a}{\sin A}$$
$$c = \frac{a \sin C}{\sin A}$$
$$c \doteq \frac{31.2 \times 0.899}{0.588}$$
$$\doteq 47.7$$

$\sin 116°$
$= \sin (180° - 116°)$
$= \sin 64°$
$= 0.899$

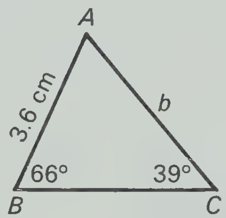


The length of c is 47.7 cm.

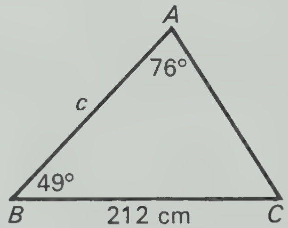
EXERCISE 8-12

B **1.** Find the indicated side in the following.

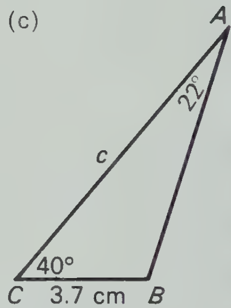
(a)



(b)



(c)

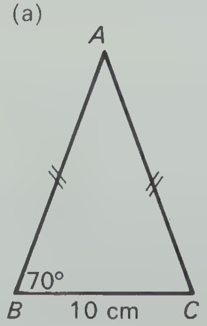


2. Use the law of sines in the following:

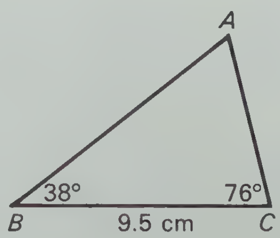
(a) In $\triangle ABC$, $\angle A = 20°$, $\angle C = 85°$, $a = 11$ cm. Find b.
(b) In $\triangle ABC$, $\angle B = 42°$, $\angle C = 69°$, $b = 8.5$ cm. Find a.
(c) In $\triangle ABC$, $\angle B = 70°$, $\angle C = 75°$, $a = 5.6$ cm. Find b.
(d) In $\triangle ABC$, $\angle A = 100°$, $\angle B = 30°$, $a = 7.8$ cm. Find c.
(e) In $\triangle ABC$, $\angle A = 20°$, $\angle B = 30°$, $a = 5.2$ cm. Find b.

3. Solve the following triangles.

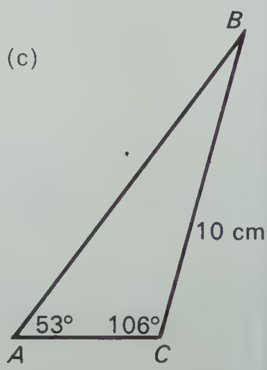
(a)



(b)

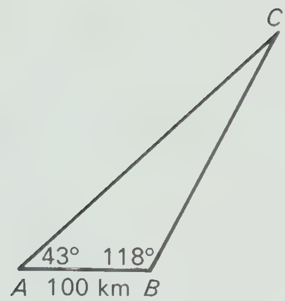


(c)

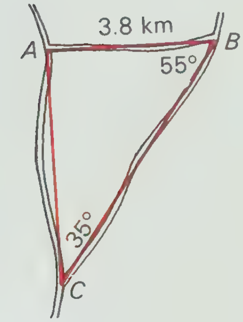


(d) $\triangle ABC$, $c = 20$ cm, $\angle B = 48°$, $\angle C = 63°$
(e) $\triangle ABC$, $a = 7.8$ cm, $\angle B = 27°$, $\angle C = 102°$

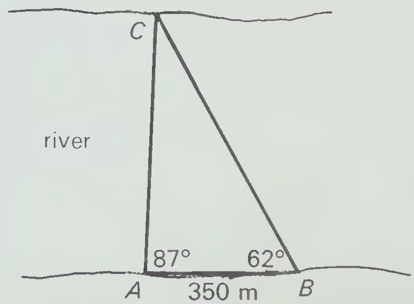
4. Town A is 100 km west of town B. The inclination of town C is 43° from A. From B, the inclination of C is 118°. Find the distances AC and BC.



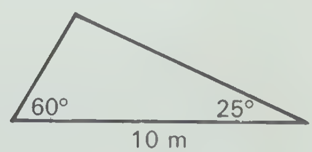
5. Two highways diverge at 35° from point C. A third road AB is 3.8 km long and joins the two highways as in the diagram, making an angle of 55° with one of the roads. How far are the A and B intersections from C?



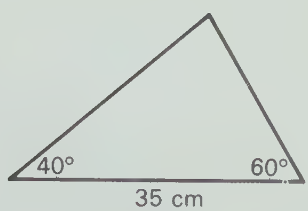
6. In order to find the distance across a river, two students measured a 350 m base line, AB, and measured the angles at A and B with respect to a point C across the river as in the diagram. Find the distance AC across the river.



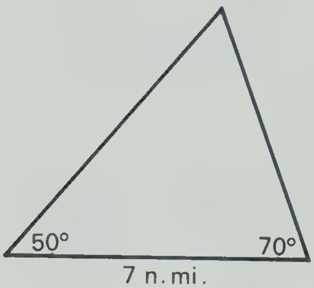
7. A greenhouse is 10 m wide and the rafters make angles of 25° and 60° with the joists. Find the length of each type of rafter.



8. A triangular plate has one side 35 cm and angles of 40° and 60° as in the diagram.
(a) Find the perimeter of the plate.
(b) Find the cost of grinding the edges at 5¢/cm.



9. From points on hilltops 17 km apart, two observers sight a third hill making horizontal angles of 41° and 37° at the hilltops, with the line joining the hills. Make a diagram and calculate the distances to the third hill.

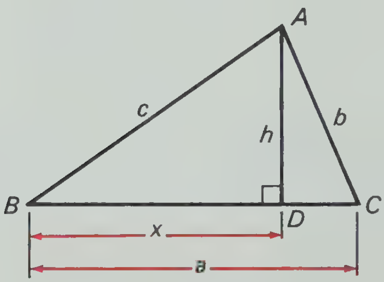


10. A ship sails past a lighthouse. From two points 7 nautical miles apart the captain sights the lighthouse at angles of 50° and 70°. How far was the ship from the lighthouse at the time of the sightings?

8.13 THE LAW OF COSINES

When an oblique triangle with two sides and the contained angle (SAS) or three sides (SSS) is given, we use the law of cosines.

(i) Acute Triangle



In $\triangle ABD$, $\frac{x}{c} = \cos B$

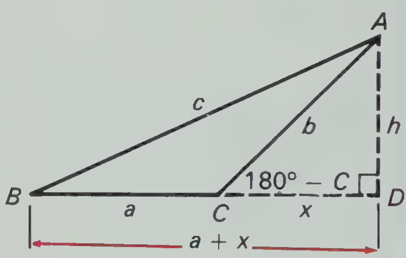
$$x = c \cos B$$

$$\text{and } c^2 = h^2 + x^2$$

In $\triangle ACD$,

$$\begin{aligned} b^2 &= h^2 + (a - x)^2 \\ &= h^2 + a^2 - 2ax + x^2 \\ &= a^2 + (h^2 + x^2) - 2ax \\ b^2 &= a^2 + c^2 - 2ac \cos B \end{aligned}$$

(ii) Obtuse Triangle



In $\triangle ACD$, $\frac{x}{b} = \cos(180° - C)$

$$\begin{aligned} x &= b \cos(180° - C) \\ &= -b \cos C \end{aligned}$$

$$\text{and } b^2 = h^2 + x^2$$

In $\triangle ABD$,

$$\begin{aligned} &= h^2 + a^2 + 2ax + x^2 \\ &= a^2 + (h^2 + x^2) + 2ax \\ &= a^2 + b^2 + 2a(-b \cos C) \\ c^2 &= a^2 + b^2 - 2ab \cos C \end{aligned}$$

Similarly

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

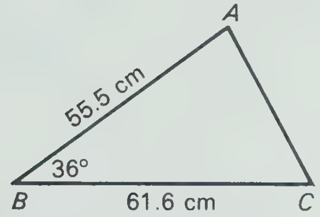
Similarly

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

EXAMPLE 1. *In $\triangle ABC$, $a = 61.6$ cm, $\angle B = 36°$, and $c = 55.5$ cm. Find b.*

Solution



$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$b^2 = (61.6)^2 + (55.5)^2 - 2(61.6)(55.5)(.809)$$

$$\doteq 3795 + 3080 - 5532$$

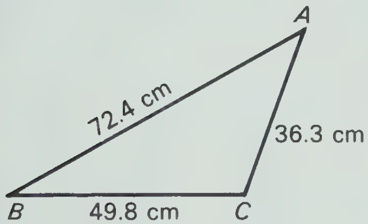
$$\doteq 1343$$

$$b \doteq \sqrt{1343}$$

$$\doteq 36.6$$

EXAMPLE 2. *In $\triangle ABC$, $a = 49.8$ cm, $b = 36.3$ cm, and $c = 72.4$ cm. Find $\angle C$.*

Solution



$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$2ab\cos C = a^2 + b^2 - c^2$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos C = \frac{(49.8)^2 + (36.3)^2 - (72.4)^2}{2(49.8)(36.3)}$$

$$\doteq \frac{2480 + 1318 - 5242}{3615}$$

$$\doteq -0.3994$$

$$\cos 66° \doteq 0.4067 \text{ and } \cos 67° \doteq 0.3907$$

$$\angle C \doteq 180° - 66°$$

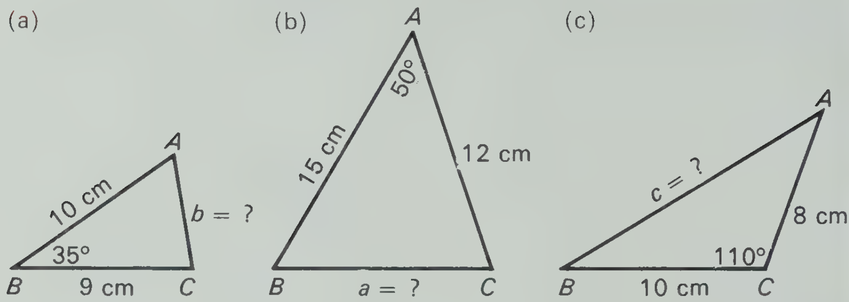
$$\doteq 114°$$

θ	$\cos\theta$
0	0.0000
1	0.9998
2	0.9994
3	0.9986
4	0.9976
5	0.9962
6	0.9945
7	0.9925
8	0.9903
9	0.9877
10	0.9848
11	0.9816
12	0.9781
13	0.9744
14	0.9703
15	0.9659
16	0.9613
17	0.9563
18	0.9511
19	0.9455
20	0.9397
21	0.9336
22	0.9272
23	0.9205
24	0.9135
25	0.9063
26	0.8988
27	0.8910
28	0.8829
29	0.8746
30	0.8660
31	0.8572
32	0.8480
33	0.8387
34	0.8290
35	0.8192
36	0.8090
37	0.7986
38	0.7880

EXERCISE 8-13

B

1. Find the indicated side in the following:



2. Use the law of cosines in the following:
(a) In $\triangle ABC$, $a = 5.53$ cm, $c = 5.13$ cm, and $\angle B = 43°$. Find b.
(b) In $\triangle ABC$, $a = 45$ cm, $c = 50$ cm, and $\angle B = 123°$. Find b.
(c) In $\triangle ABC$, $a = 6$ cm, $b = 4$ cm, $c = 5$ cm. Find $\angle A$.
(d) In $\triangle ABC$, $a = 50$ cm, $b = 46$ cm, $c = 81$ cm. Find $\angle C$.

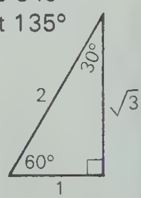
REVIEW EXERCISE

1. Find the primary trigonometric ratios of an angle, θ, in standard position if a point on the terminal arm is
(a) $(8, 15)$ (b) $(-5, 12)$ (c) $(1, -5)$ (d) $(1, -\sqrt{3})$

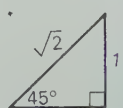
2. Find the reciprocal trigonometric ratios of an angle in standard position if a point on the terminal arm is
(a) $(12, -5)$ (b) $(-1, 1)$ (c) $(2, 7)$ (d) $(-\sqrt{3}, -1)$

3. If θ is a second quadrant angle and $\cos\theta = -\frac{4}{5}$, state the coordinates of a point on the terminal arm and find $\sin\theta$.

4. Using the tables and special relationships, find
(a) sin 250° (b) cos 150° (c) tan 220° (d) sin 130°
(e) cos 240° (f) tan 300° (g) sin 340° (h) cos 310°
(i) tan 140° (j) cos 220° (k) sin 250° (l) tan 120°
(m) csc 120° (n) tan 120° (o) csc 300° (p) sec 345°
(q) sec 160° (r) sec 220° (s) cot 110° (t) cot 135°



5. Evaluate without using tables.
(a) $\sin 30°$ (b) $\sin 60°$ (c) $\cos 30°$ (d) $\cos 60°$
(e) $\sin^2 30° + \cos^2 60°$ (f) $2 \sin 30° \cos 60°$
(g) $\dfrac{1 - \sin^2 30°}{\sin^2 60°}$ (h) $\dfrac{\cos^2 30°}{1 - \cos^2 60°}$



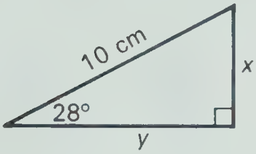
6. Evaluate without using tables.
(a) $\sin 45°$ (b) $\cos 45°$ (c) $\tan 45°$
(d) $1 + \tan^2 45°$ (e) $\dfrac{\sin 45°}{\cos 45°}$ (f) $\sec^2 45°$

7. Sketch the graphs of each of the primary trigonometric functions by plotting only the critical points for $0° \leq \theta \leq 360°$. For the sine function

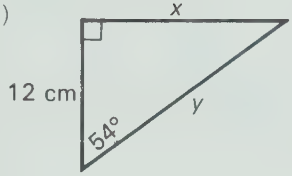
the critical points occur when θ is 0°, 90°, 180°, 270°, 360°.

8. Solve the triangles below for x and y.

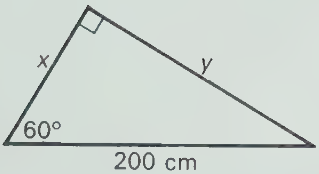
(a)



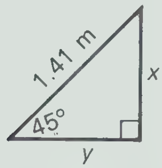
(b)



(c)

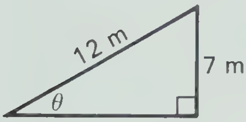


(d)

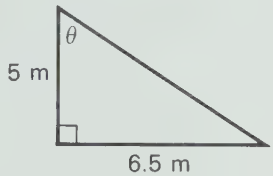


9. Find the angle θ in each of the following.

(a)

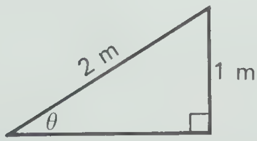


(b)



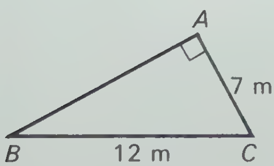
(c)

(d)

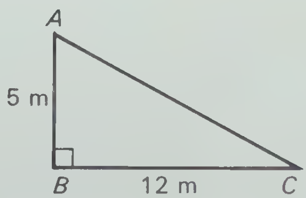


10. Solve the following triangles.

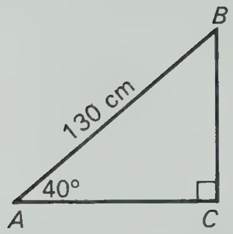
(a)



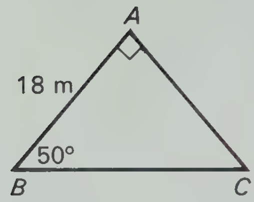
(b)



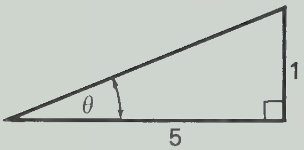
(c)



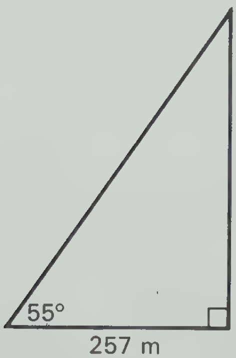
(d)



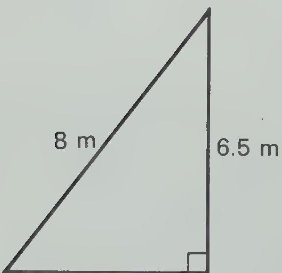
11. How high up a wall will a 7 m ladder reach if the ladder must make an angle of 65° with the ground?



12. A ramp at a football stadium rises 1 m for every 5 m of run. Calculate the angle of inclination of the ramp.



13. A tourist in New York measured the shadow of the Empire State Building to be 257 m. Calculate the height of the building to the nearest metre if the angle of elevation of the sun was 55°.



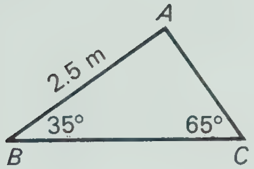
14. An 8 m ladder reaches 6.5 m up a wall.

(a) Calculate the angle the ladder makes with the wall.

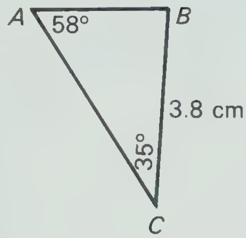
(b) How far is the foot of the ladder from the wall?

15. Solve the following oblique triangles.

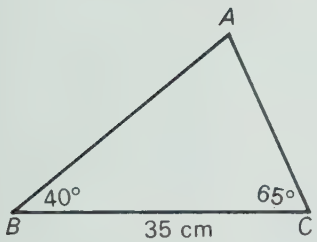
(a)



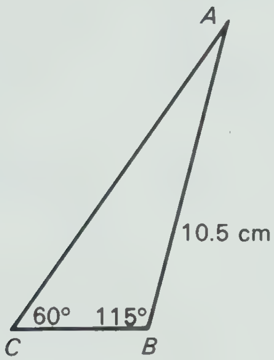
(b)



(c)



(d)

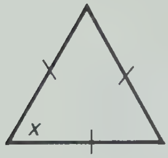


REVIEW AND PREVIEW TO CHAPTER 9

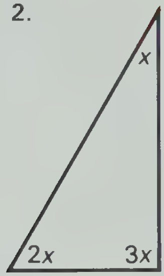
EXERCISE 1

Calculate the values of x and y.

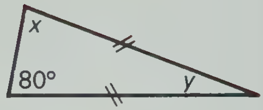
1.



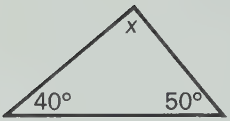
2.



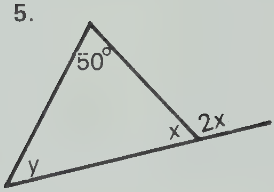
3.



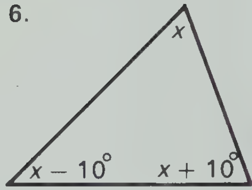
4.



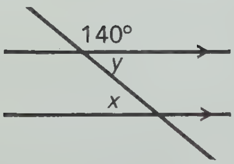
5.



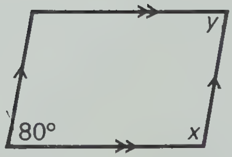
6.



7.



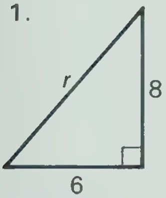
8.

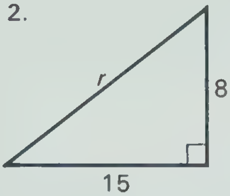


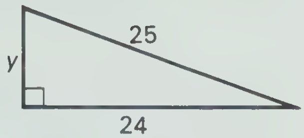
9.

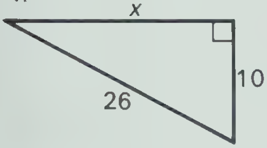
EXERCISE 2

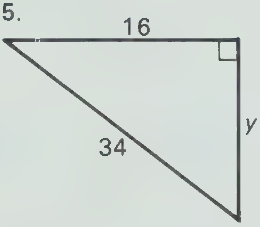
Use the theorem of Pythagoras to find the values of the variables.

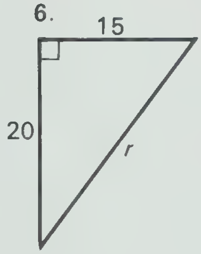
1.


2.


3.


4.


5.


6.


Solve for the indicated variable.

1. $\dfrac{x}{13.75} = \sin 46^\circ$
2. $\dfrac{x}{1.736} = \cos 66^\circ$
3. $\dfrac{x}{137.8} = \tan 49^\circ$
4. $\dfrac{x}{0.7681} = \sin 33^\circ$
5. $\dfrac{x}{11.86} = \cos 13^\circ$
6. $\sin \theta = \dfrac{7.761}{8.412}$
7. $\cos \theta = \dfrac{51.32}{66.85}$
8. $\tan \theta = \dfrac{273.4}{346.5}$
9. $\sin \theta = \dfrac{0.7132}{0.9476}$
10. $\cos \theta = \dfrac{24.73}{48.66}$

CHAPTER 9

Vectors in Two Dimensions

You have studied mathematical objects such as numbers and geometric figures. In this chapter, you will study mathematical objects that can be represented by either an ordered pair or a line segment with *direction* indicated.

9.1 DISPLACEMENTS

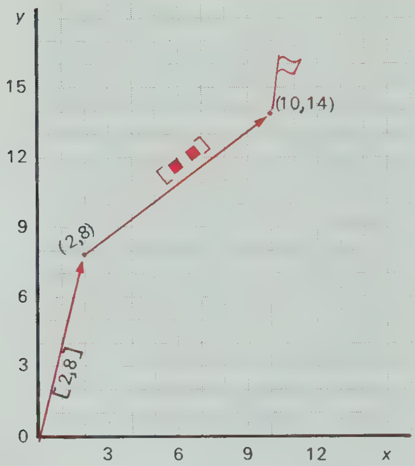
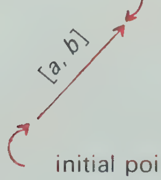


Figure 9-1



Consider one hole of a golf course related to a coordinate system so that the "tee" is at the origin and the "cup" is at the point (10, 14) as in figure 9-1. If the drive goes from the origin to the point (2, 8) then we can represent the displacement by a directed line segment. The displacement could also be represented by the ordered pair with square brackets [2, 8]. How would you represent the displacement from (2, 8) to (10,14) in the diagram?

[5, 2] represents a directed line segment which causes a displacement "five units to the right and two up." In figure 9-2 the displacement [5, 2] is shown with the initial points 0(0, 0), $A(5, 2)$, $B(3, 7)$, and $C(-5, 4)$. Note that the same displacement, [5, 2], originates from four different *initial* points to produce four different *terminal* points.

Displacements can be represented *geometrically* by directed line segments and *algebraically* by ordered pairs with square brackets [*a*, *b*] where *a* and *b* are the *components* of the displacement.

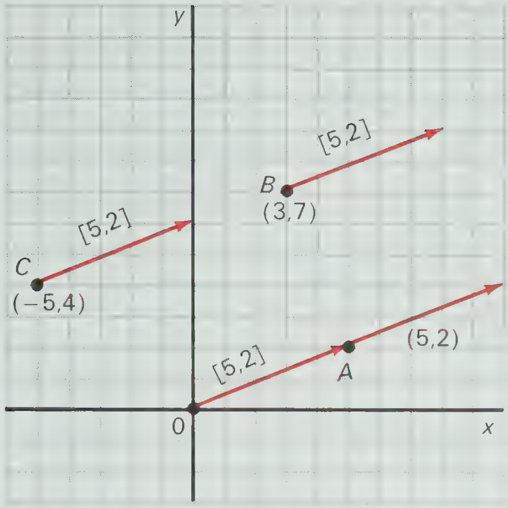


Figure 9-2

Starting with the word "math" and changing one letter at a time to form a new word, can you reach "game" in three changes?

EXERCISE 9-1

A **1.** Find $[a, b]$, $[c, d]$, $[e, f]$, and $[g, h]$, in figure 9-3.

2. In figure 9-4, find the terminal points determined by the displacement [2, 5] for each of the initial points 0(0, 0), $A(0, 6)$, $B(1, -2)$, and $C(-3, -2)$.

3. Find the displacement determined by the initial point (3, 2) for each of the terminal points in figure 9-5.

(a) 0(0, 0) (b) $A(2, 7)$ (c) $B(8, 10)$
(d) $C(3, -3)$ (e) $D(8, 0)$

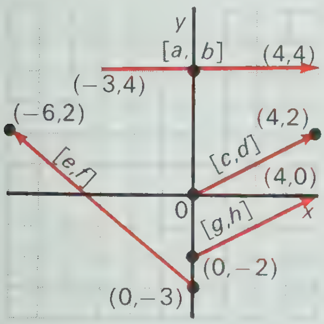


Figure 9-3

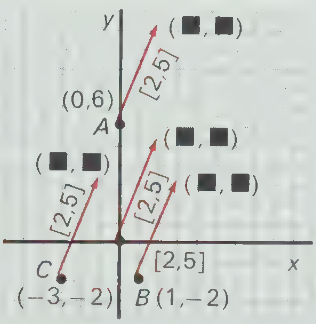


Figure 9-4

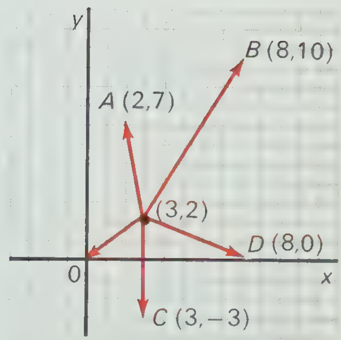


Figure 9-5

B **4.** Make a diagram to show each of the following displacements as a directed line segment and label (i) initial point, (ii) terminal point, (iii) the displacement as an ordered pair $[a, b]$.

From:

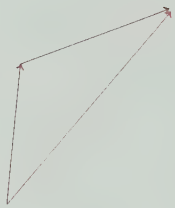
(a) (0, 0) to (3, 6) (b) (−2, 5) to (10, 8)
(c) (−4, 0) to (6, −3) (d) (2, 1) to (5, −4)
(e) (5, −4) to (2, 1)

5. On the same diagram, show the terminal points determined by the

displacement [4, 3] for each of the initial points:
(a) (2, 5) (b) (−4, −3) (c) (0, −2) (d) (3, 8) (e) (−5, −2)

6. Each of the following pairs of points describes a displacement. Make a diagram and determine which are the same displacement.

(a) from (0, 0) to (3, 4) (b) from (−3, 4) to (1, 7)
(c) from (7, 0) to (11, 3) (d) from (6, 3) to (10, 5)
(e) from (5, 1) to (1, −2) (f) from (−4, 3) to (0, 0)

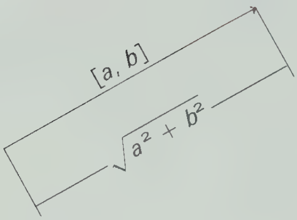


C **7.** Choosing any initial point, plot the following displacements on squared paper and find a single displacement equivalent to each.

(a) [3, 2] followed by [2, 7] (b) [5, 4] followed by [−2, 3]
(c) [6, 4] followed by [−2, −5] (d) [4, 3] followed by [4, 3]

8. Find the lengths of the directed line segments indicated by the following displacements:

(a) [−4, 3] (b) [5, 12] (c) [−5, −12]
(d) [8, −15] (e) [−7, 24] (f) [−15, −8]
(g) [−6, 8] (h) [−3, −4]



9.2 VECTORS

Mathematical objects such as displacements, which can be represented by directed line segments or by ordered pairs with square brackets, we shall call vectors. The vector from A to B shown in figure 9-6 can be represented geometrically by the directed line segment $\overrightarrow{AB}$, or algebraically by the ordered pair [6, 4]. The geometric vector $\overrightarrow{AB}$ can also be represented simply by $\overrightarrow{v}$.

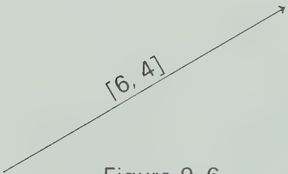


Figure 9-6

EXERCISE 9-2

B **1.** For each of the following make a diagram on squared paper and state vector $\overrightarrow{AB}$ in the form $[a, b]$.

(a) B is two units to the right and four units above A.
(b) B is three units to the left and five units up from A.
(c) A is three units to the left of B.
(d) A is five units above B.
(e) A and B occupy the same position.

In each of the above, $\overrightarrow{AB}$ (read vector AB) determines the position of B relative to A.

2. Plot the points $A(2, 4)$, $B(7, 2)$, $C(-3, -6)$, $D(5, -1)$ and find the algebraic vectors represented by $\overrightarrow{AB}$, $\overrightarrow{BA}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DA}$, $\overrightarrow{AC}$, $\overrightarrow{DB}$.

3. (a) Plot the points $P(-1, 3)$, $Q(-3, -5)$, $R(8, 1)$, $S(10, 9)$, and draw the vectors represented by $\overrightarrow{PQ}$, $\overrightarrow{QR}$, $\overrightarrow{SR}$, $\overrightarrow{PS}$. Identify figure $PQRS$.
(b) Find the algebraic vectors represented by $\overrightarrow{PQ}$, $\overrightarrow{QR}$, $\overrightarrow{SR}$, $\overrightarrow{PS}$.

4. Three aircraft are flying northeast at 125, 150, and 200 knots respectively. Using a suitable scale, draw vectors to show the flight paths of the aircraft.

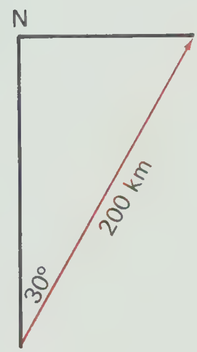
(a) How are the three vectors related?
(b) How do they differ?

(c) How would you change the vector of the first aircraft so that it has the same vector as the second aircraft?

5. (a) Using a suitable scale, draw vectors to represent a flight path 200 nautical miles north followed by 400 nautical miles west.
(b) Draw the vector to show the resulting displacement from the starting point to the terminal point of the flight.
(c) Measure the length of this new vector and the angle it makes with the horizontal vector.

6. A man rows a boat at the rate of 5 knots in a direction directly across a river. The river has a current of 12 knots.
(a) Using a suitable scale, make a vector diagram to show the actual direction of his travel.
(b) Find his actual speed:
(i) by measurement. (ii) by calculation.

7. An aircraft flies 200 km in a direction 30° east of north. Use trigonometry to express this vector in the form [a, b].



8. Express the following as vectors in the form [a, b].
(a) 500 km north (b) 100 km east
(c) 200 km in a direction 120° west of north

9. (a) Draw a geometric vector $\overrightarrow{PQ}$ with magnitude 10 units at an angle of 30° to the horizontal measured counterclockwise. Draw $\overrightarrow{QR}$ with magnitude 5 units and direction 90° to the horizontal measured clockwise.
(b) Find the direction of $\overrightarrow{PR}$ and the approximate magnitude.

9.3 EQUAL VECTORS

The algebraic vector [5, 3] is represented geometrically by a line segment with an arrow to show direction: "five units to the right and up three units." Figure 9-7 shows the vector [5, 3] represented geometrically by selecting several initial points. While each geometric vector in the figure has a different initial point, each can be represented by the same ordered pair [5, 3]. The magnitude or length of each vector is

$$\sqrt{5^2 + 3^2} = \sqrt{34}$$

and each has the same slope, $\frac{3}{5}$.

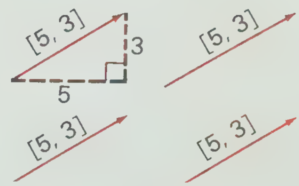
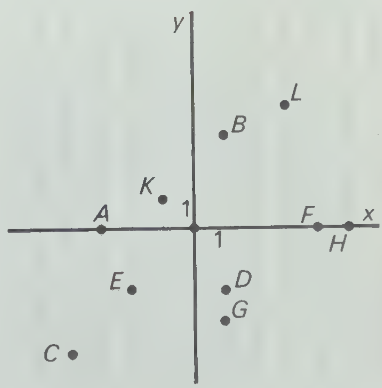


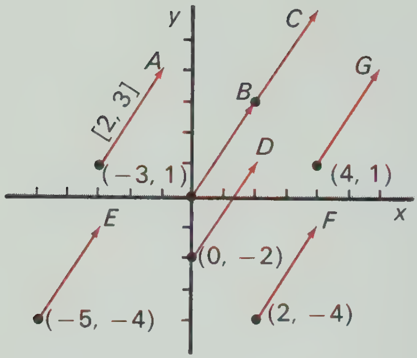
Figure 9-7

Vectors with the same magnitude and direction can be represented by the same ordered pair and are called equal vectors.

EXERCISE 9-3

A 1. (a) State the algebraic vectors represented by each of the following.
(i) $\overrightarrow{AB}$
(ii) $\overrightarrow{CD}$
(iii) $\overrightarrow{EF}$
(iv) $\overrightarrow{GH}$
(v) $\overrightarrow{KL}$
(b) Name the vectors in (a) which are equal vectors.





B **2.** Draw three geometric vectors representing each of the algebraic vectors.
(a) [2, 6] (b) [−4, 3] (c) [0, −5] (d) [1, 1] (e) [−5, −7]

3. Find the missing terminal points if all the vectors are equal vectors.

4. If $A(-2, 5)$, $B(3, 2)$, and $C(1, 6)$ are three points in the plane, make a diagram and find the coordinates of point D so that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal vectors.

5. Given points $A(2, 7)$, $B(3, 1)$, and $C(-2, 0)$, find the following.
(a) The coordinates of D so that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal vectors.
(b) The coordinates of point E so that $\overrightarrow{AC} = \overrightarrow{BE}$.

Test your skill:
150% of 30

6. (a) Given points $P(-5, 5)$, $Q(9, 2)$, and $R(4, -7)$, find the coordinates of a point S so that $PQRS$ is a parallelogram.
(b) Verify that $PQRS$ is a parallelogram by showing that $\overrightarrow{PQ} = \overrightarrow{SR}$.
(c) Verify that $PQRS$ is a parallelogram by showing that $\overrightarrow{RQ} = \overrightarrow{SP}$.
(d) Is it necessary to show both $\overrightarrow{PQ} = \overrightarrow{SR}$ and $\overrightarrow{RQ} = \overrightarrow{SP}$ in order to verify that $PQRS$ is a parallelogram?

7. (a) Complete the following table

Initial point	Terminal point	Algebraic vector $[a, b]$	Magnitude $\sqrt{a^2 + b^2}$	Slope $\frac{b}{a}$
$A(2, 4)$	$B(3, 6)$	[1, 2]	$\sqrt{5}$	2
$C(2, 7)$	$D(5, 3)$			
$E(-2, 1)$	$F(0, 5)$			
$G(3, -2)$	$H(6, -6)$			
$I(-4, -3)$	$J(-3, -1)$			
$K(0, -5)$	$L(1, -3)$			
$M(0, 2)$	$N(3, -2)$			
$P(5, 3)$	$Q(4, 1)$			

(b) Name vectors from part (a) which are equal vectors.

Note: When two vectors have the same magnitude and the same slope, they are not equal unless they can be represented by the same ordered pair.

8. (a) If [5, 3], [5, *b*], and [*a*, 3] are equal vectors, state the numerical values of *a* and *b*.
(b) If $[a, b] = [c, d]$, how are the numbers *a*, *b*, *c*, and *d* related?
(c) If $a = c$ and $b = d$, how are $[a, b]$ and $[c, d]$ related?
In general:

> two vectors $[a, b]$ and $[c, d]$ are equal if and only if $a = c$ and $b = d$.

9.4 ADDITION OF VECTORS

When we add two vectors, the result is another vector. In the investigation which follows you will develop a method for adding vectors.

INVESTIGATION 9.4

1. An aircraft flies 200 km east from town A to town B, then continues another 300 km east to town C.
(a) Make a scale drawing to show the two vectors, $\overrightarrow{AB}$ and $\overrightarrow{BC}$.
(b) How are $\overrightarrow{AC}$ and $(\overrightarrow{AB} + \overrightarrow{BC})$ related?
(c) Express $\overrightarrow{AB}$, $\overrightarrow{BC}$, $(\overrightarrow{AB} + \overrightarrow{BC})$ and $\overrightarrow{AC}$ as ordered pairs and verify the result obtained in (b).

2. A man walks 200 m south from point *P* to point *Q*. At *Q* he changes direction and walks 500 m north to point *R*.
(a) Make a scale drawing to show the two vectors.
(b) Draw $\overrightarrow{PR}$ to show $\overrightarrow{PQ} + \overrightarrow{QR}$.
(c) How are $\overrightarrow{PR}$ and $(\overrightarrow{PQ} + \overrightarrow{QR})$ related?
(d) Express $\overrightarrow{PR}$, $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ as ordered pairs and verify the result obtained in (c).

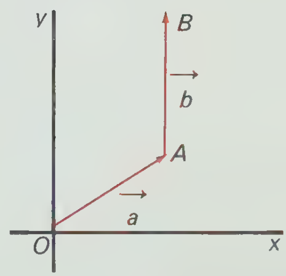
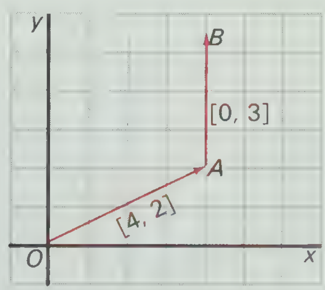


Figure 9-8

3. A ship sails from point *O* to point *A*, represented by $\vec{a}$, followed by a trip from point *A* to point *B*, represented by $\vec{b}$ as shown in figure 9-8.
(a) Name a single vector which is the result of a move from *O* to *A* followed by a move from *A* to *B*.
(b) How is the vector found in (a) related to $\vec{a}$ and $\vec{b}$?
(c) Make a diagram to show the relationship between $\vec{a}$, $\vec{b}$ and $(\vec{a} + \vec{b})$.
(d) Identify the figure formed by $\vec{a}$, $\vec{b}$ and $(\vec{a} + \vec{b})$.

4. Suppose the ship in question 3 had sailed according to the vector [4, 2] followed by the vector [0, 3].
(a) Name a single vector, in the form [*a*, *b*], which is the result of the move from *O* to *A* followed by the move from *A* to *B*.
(b) How is the vector found in (a) related to [4, 2] and [0, 3]?
(c) Make a diagram to show the relationship between [4, 2], [0, 3] and



$[4, 2] + [0, 3]$.

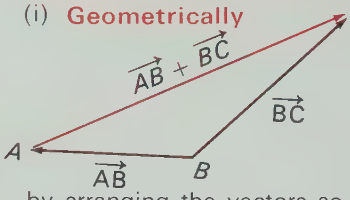
(d) Identify the figure formed by the three vectors.

5. (a) Find $[-2, 5] + [7, -3]$ and make a diagram to check your result.

(b) Find $[-2, 5] + [3, 4] + [4, -3]$ and make a diagram to check your result.

In general vectors can be added

(i) **Geometrically**



by arranging the vectors so that we can join the *initial* point of the first vector to the *terminal* point of the last vector.

(ii) **Algebraically**

$[a, b] + [c, d]$
$= [a + c, b + d]$

by adding the respective first components and second components.

Continue the pattern:

6. (a) Given $\overrightarrow{AB} = [5, 1]$, find the ordered pair which represents $2\overrightarrow{AB}$ by finding $\overrightarrow{AB} + \overrightarrow{AB}$.

(b) How are the components of the vector determined by $2\overrightarrow{AB}$ related to the components of $\overrightarrow{AB}$?

(c) How are the vectors $\vec{a} = [5, 1]$ and $\vec{b} = [15, 3]$ related?

7. (a) Find $[3, 7] + [-1, -4]$ and identify the result of adding these vectors.

(b) Find $[a, b] + [c, d]$ and identify the result of adding any two vectors.

(c) Is the sum of two vectors always a vector? Name the property.

Simplify the left side and right side separately in question 8 and 9 then insert the proper sign, $=$ or $\neq$.

8. (a) $[-3, 7] + [2, -5] \ \boxed{?} \ [2, -5] + [-3, 7]$

(b) $[a, b] + [c, d] \ \boxed{?} \ [c, d] + [a, b]$

(c) Is addition of vectors commutative?

9. (a) $([2, 4] + [3, -2]) + [-4, 8] \ \boxed{?} \ [2, 4] + ([3, -2] + [-4, 8])$

(b) $([a, b] + [c, d]) + [e, f] \ \boxed{?} \ [a, b] + ([c, d] + [e, f])$

(c) Is addition of vectors associative?

10. (a) Find the sums:

(i) $[4, 3] + [0, 0]$ (ii) $[0, 0] + [-7, -2]$

(iii) $[-4, 2] + [0, 0]$ (iv) $[0, 0] + [-3, -5]$

(b) Add

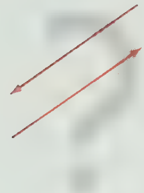
(i) $[a, b] + [0, 0]$ (ii) $[0, 0] + [a, b]$

(c) Find values of x and y so that

(i) $[a, b] + [x, y] = [a, b]$

(ii) $[x, y] + [a, b] = [a, b]$

(d) What is the identity or zero vector?



11. (a) Find the result of a move "five to the right and three up" followed by "five to the left and three down."
(b) Add the vectors [7, 4] + [−7, −4] and compare this result to part (a).

12. Find [x, y] in each case.
(a) [2, 3] + [x, y] = [0, 0] (b) [x, y] + [4, 11] = [0, 0]
(c) [−4, 7] + [x, y] = [0, 0] (d) [−3, −5] + [x, y] = [0, 0]
(e) [a, b] + [x, y] = [0, 0]

In general,

> The inverse, or negative of [a, b] is [$-a$, $-b$].
> $-[a, b] = [-a, -b]$

13. (a) Make a diagram to show $\vec{a}$ = [2, 5] and its negative, both having the same initial point.
(b) Repeat part (a) for
(i) $\vec{b}$ = [−3, 2] (ii) $\vec{c}$ = [−5, −7] (iii) $\vec{d}$ = [4, −3]
(c) From your diagrams in (a) and (b) state how a vector and its negative are
(i) similar (ii) different

EXERCISE 9-4

A **1.** Add the following vectors.
(a) [3, 5] + [4, 6] (b) [−2, 7] + [6, 1]
(c) [−3, −5] + [4, 6] (d) [2, −6] + [3, 1]
(e) [−2, −3] + [−3, −4] (f) [2, 6] + [2, 6]
(g) [4, 7] + [−4, −7] (h) [11, 1] + [−1, −11]
(i) [0, 0] + [2, 5] (j) [6, 3] + [2, 7] + [−5, −4]

2. Find [x, y] for each of the following.
(a) [x, y] + [3, −7] = [0, 0] (b) [2, −6] + [x, y] = [0, 0]
(c) [x, y] + [−18, 4] = [−18, 4] (d) [3, 5] + [−2, 6] = [x, y]
(e) [4, −3] + [x, y] = [4, −3] (f) [x, y] + [x, y] = [6, 10]

B **3.** Using a suitable scale, find the sums of the following vectors geometrically. Approximate the magnitude and direction by measurement.
(a) 5 km east followed by 12 km north
(b) 30 km north followed by 15 km south
(c) 14 km northwest followed by 3 km west

4. Verify each of the following by simplifying the left and right sides separately.
(a) [3, 7] + [−2, 5] = [−2, 5] + [3, 7]
(b) ([2, 1] + [4, 3]) + [−2, 6] = [2, 1] + ([4, 3] + [−2, 6])

5. Find the following sums geometrically using a vector diagram.
(a) [3, 2] + [2, 5] (b) [4, 1] + [5, 3]
(c) [3, −4] + [1, 5] (d) [−4, −2] + [5, 3]
(e) [2, 7] + [−3, −7] (f) [−3, −4] + [−5, 2]

6. Find the following sums geometrically, using a vector diagram.

(a) $[3, -2] + [2, -3] + [-4, 8]$
(b) $[-3, -2] + [0, 5] + [-3, -2] + [5, 0]$

9.5 MULTIPLICATION OF A VECTOR BY A SCALAR

In Section 9.2 vectors were represented as directed line segments or as ordered pairs with square brackets $[a, b]$. When working with vectors, we can also use real numbers, which we shall call scalars. What is the effect of multiplying a vector by a scalar?

INVESTIGATION 9.5

1. Two ships sail from the same port so that ship A sails according to the vector $\vec{a} = [5, 12]$ and ship B according to $\vec{b} = [10, 24]$.

(a) Make a diagram to show these vectors in geometric form.
(b) From your diagram, find vector $\vec{c}$ so that $\vec{a} + \vec{c} = \vec{b}$.
(c) How are $\vec{a}$ and $\vec{c}$ related?
(d) How is the length of $\vec{b}$ related to the length of $\vec{a}$? Find the lengths.
(e) In algebra, $x + x = 2x$. Express a similar result for $\vec{a} + \vec{a}$.
(f) State the relationship between $\vec{b}$ and $\vec{a}$.

2. An aircraft flies according to the vector $\vec{a} = [3, 4]$.

(a) Find $\vec{b}$ so that $\vec{b} = 3\vec{a}$.
(b) Find the length of $\vec{b}$ and the length of $\vec{a}$.
(c) How are the lengths of $\vec{a}$ and $\vec{b}$ related?

$\vec{O}$ is the ZERO vector $[0, 0]$

3. Given $\vec{a} = [7, 9]$.

(a) Find $\vec{b} = (-1)\vec{a}$. $(-1)\vec{a}$ is usually written $-\vec{a}$.
(b) Compare:
  (i) The magnitudes of $\vec{a}$ and $\vec{b}$.
  (ii) The directions of $\vec{a}$ and $\vec{b}$.
(c) How are $\vec{a}$ and $\vec{b}$ related?
(d) Find $(\vec{a}) + (-\vec{a})$.

4. Given vector $\vec{a} = [3, 2]$.

(a) Find $2\vec{a}$, $3\vec{a}$, $-\vec{a}$, $-3\vec{a}$ (b) Draw $\vec{a}$, $2\vec{a}$, $3\vec{a}$, $-\vec{a}$, $-3\vec{a}$

5. Given $\vec{v} = [a, b]$ and the scalar x, find:

(a) $x[a, b]$ (b) $2x[a, b]$
(c) $-x[a, b]$ (d) $\frac{1}{2}x[a, b]$

6. State the effect on: (i) magnitude; and (ii) direction, of multiplying the vector $[p, q]$ by each of the following scalars.

(a) 3 (b) -2 (c) $\frac{1}{4}$
(d) A positive number k (e) A negative number k

In general,

$$\boxed{k[x, y] = [kx, ky]}$$

Test your skill: $(-1)^{112}$

EXERCISE 9-5

A 1. State each of the following in the form $[a, b]$.

(a) $2[4, 7]$ (b) $3[6, -1]$ (c) $\frac{1}{2}[6, 12]$
(d) $-4[-2, -5]$ (e) $\frac{2}{3}[-3, 6]$ (f) $\frac{1}{2}[0, 0]$
(g) $4[0, -2]$ (h) $-\frac{1}{4}[4, -16]$ (i) $7[3, 6]$

B 2. Simplify each of the following and express your answer in the form $[a, b]$.

(a) $3[4, 1] + 2[3, 4]$ (b) $2[-1, 5] + [3, -2]$
(c) $\frac{1}{2}[0, 4] + 4[1, 0]$ (d) $3[1, 1] + (-2)[2, 2]$
(e) $(-5)[2, 7] + (-3)[7, -2]$ (f) $3([2, 3] + [4, -7])$
(g) $4([3, 5] + 2[-1, 2])$ (h) $5([4, 2] + (-2)[2, 1])$
(i) $\frac{1}{2}(4[3, 2] + (-2)[-4, -2] + [5, -1])$

3. If $\vec{a} = [x, y]$ and $\vec{b} = [2x, 3y]$, find:

(a) $4\vec{a} + 2\vec{b}$ (b) $3\vec{a} - \vec{b}$, where $-\vec{b} = (-1)\vec{b}$
(c) $4(\vec{a} + \vec{b})$

4. Make a diagram to verify that:

(a) $3\vec{a} + 2\vec{a} = (3 + 2)\vec{a}$, for $\vec{a} = [2, -1]$
(b) $-2(\vec{a} + \vec{b}) = -2\vec{a} + (-2)\vec{b}$ for $\vec{a} = [3, 5]$ and $\vec{b} = [-2, 3]$

9.6 SUBTRACTION OF VECTORS

Subtraction of vectors is defined as the addition of the negative of the given vector. To subtract $[3, -5]$, one simply adds $[-3, +5]$.

EXAMPLE 1. *Simplify $[7, 5] - [3, -5]$.*

Solution

$$\begin{aligned}[7, 5] - [3, -5] &= [7, 5] + [-3, +5]\\ &= [4, 10]\end{aligned}$$

This result may be generalized to

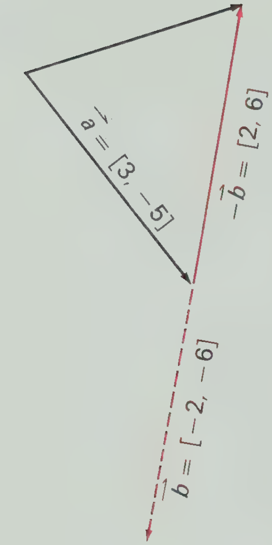
$$\boxed{[a, b] - [p, q] = [a - p, b - q]}$$

EXAMPLE 2. *Given that $\vec{a} = [3, -5]$ and $\vec{b} = [-2, -6]$, find $\vec{a} - \vec{b}$:*
(i) geometrically
(ii) algebraically

Solution (i) To subtract $[-2, -6]$, add $[+2, +6]$:

$$-[-2, -6] = [+2, +6]$$

Step 1: Draw $\vec{a} = [3, -5]$
Step 2: Draw $-\vec{b} = [2, 6]$
Step 3: Draw the resultant $\vec{a} - \vec{b} = [5, 1]$



Solution (ii)

$$\begin{aligned}\vec{a} - \vec{b} &= [3, -5] - [-2, -6]\\ &= [3 - (-2), -5 - (-6)]\\ &= [3 + 2, -5 + 6]\\ &= [5, 1]\\ \text{or, } \vec{a} - \vec{b} &= \vec{a} + (-\vec{b})\\ &= [3, -5] + [2, 6]\\ &= [5, 1]\end{aligned}$$

EXERCISE 9-6

A

1. Simplify and express your answer in the form $[a, b]$.

(a) $[4, 2] - [3, 0]$ (b) $[6, -5] - [-3, 4]$
(c) $[-3, -2] - [-7, -11]$ (d) $[5, -7] - [8, -5]$
(e) $[3, 2] - [3, 2]$ (f) $[5, 12] - [-5, -12]$
(g) $[5, 9] - [0, 0]$ (h) $[a, 6] - [0, 3]$
(i) $[5a, 11b] - [2a, 8b]$ (j) $[x, y] - [c, d]$

B

2. Simplify and express your answer in the form $[a, b]$.

(a) $3[5, -2] - 7[1, -1]$ (b) $4[6, -3] - 2[5, 4]$
(c) $\frac{1}{2}[-6, 4] - \frac{1}{4}[-4, 8]$ (d) $5[\frac{3}{5}, \frac{1}{5}] - 4[6, -1]$
(e) $-2[4, -1] - 3[1, -2]$ (f) $-5([6, -3] - [-2, 5])$
(g) $3(-2[1, 5] - 4[-1, -1])$ (h) $(4[2, 7] - [2, 7])$

Test your skill:
$(-1)^{-1}$

3. Make a diagram to show that:

(a) $3([2, 4] - [3, 7]) = 3[2, 4] - 3[3, 7]$
(b) $2(3[1, 4] - 2[-2, 1]) = 6[1, 4] - 4[-2, 1]$

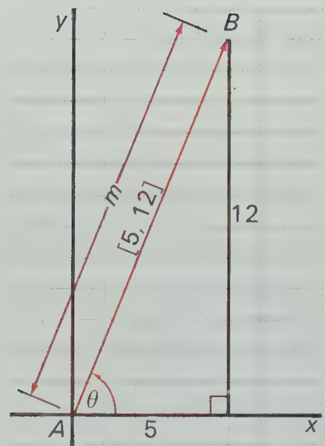
4. Verify both parts of question 3 by simplifying the left and right sides separately.

5. If $\vec{v} = [a, b]$, find $\vec{v} - \vec{v}$.

6. Express in the form $[a, b]$.

(a) $[5, 7] - [2, 4]$ (b) $[-3, 12] - [2, -7]$
(c) $3[4, -7] - 2[1, 5]$ (d) $4[3, 6] - [7, 4]$
(e) $[2, 7] - 3[-1, 4]$ (f) $[3, -7] - 4[0, 2]$
(g) $3[2, -5] - 3[-5, 2]$ (h) $[7, -2] - [2, 5] - [-3, -3]$
(i) $2[4, 1] - 3[1, -2] + 4[2, 7]$
(j) $3[-7, -1] - 2[-1, -7] - \frac{1}{2}[4, -6]$

9.7 VECTORS WITH TRIGONOMETRY



We have described vectors using ordered pairs with square brackets $[a, b]$. The magnitude and direction of a vector can also be stated using a real number, m, and the direction angle, θ, with reference to an axis.

EXAMPLE 1. *Find the magnitude, m, and direction angle, θ, for $\overrightarrow{AB} = [5, 12]$.*

Solution

$$m = \sqrt{5^2 + 12^2} \qquad \tan\theta = \tfrac{12}{5}$$

$= \sqrt{25 + 144}$ $\qquad\qquad$ $= 2.4$

$= \sqrt{169}$ $\qquad\qquad$ $\theta \doteq 67°$

$= 13$

The magnitude of $\overrightarrow{AB}$ is 13 and the direction angle is 67°.

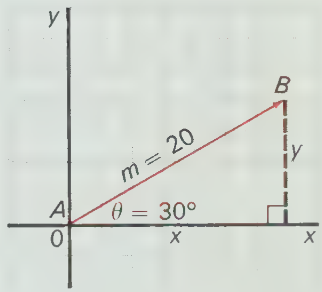
The direction angle (often called the angle of inclination) is the angle of rotation from the positive x-axis.

> In general, if $\overrightarrow{PQ} = [p, q]$, then
>
> magnitude: $m = \sqrt{p^2 + q^2}$ and
>
> direction angle: $\tan \theta = \dfrac{q}{p}$

$\tan \theta = \dfrac{y}{x}$

$\sin (180° - \theta) = \sin \theta$
$\cos (180° - \theta) = -\cos \theta$

EXAMPLE 2. *If $\overrightarrow{AB}$ has magnitude $m = 20$ and direction angle $\theta = 30°$, express $\overrightarrow{AB}$ in the form $[x, y]$.*



Solution

(i) $\dfrac{x}{20} = \cos 30°$

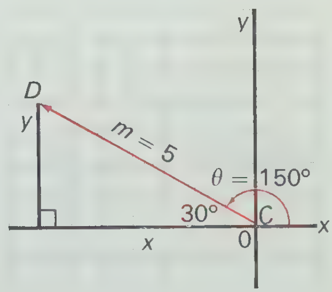
$x = 20 \cos 30°$
$\doteq 20 \times 0.87$
$\doteq 1.740$

(ii) $\dfrac{y}{20} = \sin 30°$

$y = 20 \sin 30°$
$= 20 \times 0.50$
$= 10.0$

$\overrightarrow{AB} = [1.74, 10.0]$

EXAMPLE 3. *If CD has magnitude $m = 5$ and direction angle $\theta = 150°$ express CD in the form $[x, y]$.*



Solution

(i) $\dfrac{x}{5} = \cos 150°$

$x = 5 \cos 150°$
$= 5 \times (-\cos 30°)$
$\doteq 5 \times (-0.866)$
$\doteq -4.33$

(ii) $\dfrac{y}{5} = \sin 150°$

$y = 5 \sin 150°$
$= 5 (\sin 30°)$
$= 5 \times (0.500)$
$= 2.50$

$\overrightarrow{CD} = [-4.33, 2.50]$

> A vector with magnitude, m, and direction angle, θ, can be expressed algebraically as
>
> $[m \cos \theta, m \sin \theta]$

EXERCISE 9-7

B **1.** For each of the following, find: (i) the magnitude m; and (ii) the direction angle θ.

(a) [5, 9] (b) [2, −9] (c) [−3, 4]
(d) [−10, −10] (e) [10, 17] (f) [−17, 10]
(g) [11, −11] (h) [10, 0]

2. For each of the following: (i) make a diagram; (ii) find the horizontal and vertical components; (iii) express each vector in the form $[a, b]$.

(a) $m = 12, \theta = 25°$ (b) $m = 100, \theta = 205°$
(c) $m = 1, \theta = 10°$ (d) $m = 0, \theta = 15°$
(e) $m = 2, \theta = 120°$ (f) $m = 25, \theta = 225°$
(g) $m = 10, \theta = 180°$ (h) $m = m, \theta = \theta$

3. Express each of the following vector quantities in the form $[a, b]$.

(a) a velocity of 5 m/s toward the east.
(b) a velocity of 150 knots in a northwest direction.
(c) a velocity of 500 knots from the south.
(d) a magnitude of 20 units, angle of inclination 68°.
(e) a magnitude of 111 units, angle of inclination 330°.
(f) a displacement of 6 m, angle of inclination 210°.

4. A ship sails 100 km east, then 173 km north.

(a) Express these displacements as vectors in the form $\vec{a} = [p, q]$ and $\vec{b} = [c, d]$.
(b) Find $[p, q] + [c, d]$.
(c) Check your answer to (b) by finding $\vec{a} + \vec{b}$ using another method.

5. Given two vectors, $\vec{a}$ having magnitude 10 units and direction to the right along the horizontal, and $\vec{b}$ having magnitude 8 units and angle of inclination 50°:

(a) Express $\vec{a}$ and $\vec{b}$ as ordered pairs (b) Find $\vec{a} + \vec{b}$
(c) Find $2\vec{a}$ (d) Find $\vec{a} - \vec{b}$
(e) Using a suitable scale, make a diagram to illustrate $\vec{a} + \vec{b}$, $2\vec{a}$ and $\vec{a} - b$.

6. The velocity of a ship can be represented algebraically by the vector [20, 40].

(a) Draw the geometric vector representing [20, 40].
(b) Calculate the magnitude m and the direction angle θ for the vector [20, 40].

Multiplication:
1abcde
$\underline{\quad 3}$
abcde1

7. An airplane has an air speed of 250 knots in a direction 40° east of north. Express this vector as an ordered pair.

8. A jet flying northeast with an air speed of 600 knots encounters a 60 knot wind.

(a) Express the air speed of the aircraft in the form $[a, b]$.
(b) Using algebraic vectors, express the velocity of the aircraft relative to the ground if the wind is a tail wind.
(c) Repeat (b) if the wind is a head wind.

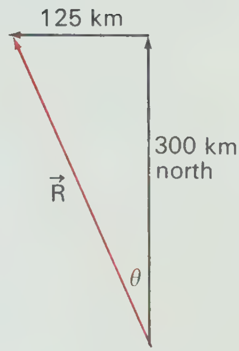
9. Express $\vec{a} + \vec{b}$ in the form $[x, y]$, given that $\vec{a}$ has magnitude 2 units at an angle of 30° to the horizontal and $\vec{b}$ has magnitude 4 units at 150° to the horizontal.

9.8 PROBLEMS SOLVED USING VECTORS

Vectors have been used to represent displacements, forces, and velocities. In general, a vector can be used to represent any quantity that has a magnitude and direction. In some of the questions in previous exercises, we have seen some applications of vectors to physical problems. In this section, further examples of the use of vectors will be studied.

EXAMPLE 1. *A ship sails 300* km *north and 125* km *west, then develops engine trouble. Find the distance and direction in which a rescue ship must sail to go directly to the first ship if both ships leave from the same point.*

Solution Let the vector which represents the course of the rescue ship be represented by R and let θ represent the direction of R relative to north. The magnitude of R is



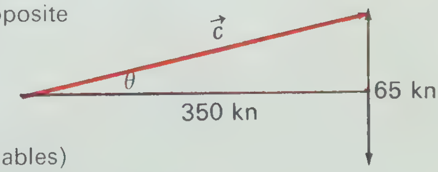
$$|\vec{R}| = \sqrt{300^2 + 125^2}$$
$$= \sqrt{90\,000 + 15\,625}$$
$$= \sqrt{105\,625} = 325$$
$$\tan\theta = \tfrac{125}{300} \doteq 0.4166$$
$$\theta \doteq 23° \qquad \text{(from tables)}$$

The rescue ship must sail 325 km in a direction 23° west of north.

EXAMPLE 2. *A pilot wishes to fly his aircraft according to the vector given by 350 knots east. A 65 knot north wind causes the pilot to alter his course. Make a vector diagram and calculate the direction and air speed of the course.*

Solution The pilot must set a course, $\vec{c}$, which is the sum of the two vectors representing the actual flight (350 knots, east) and the opposition to the wind. Note that a north wind blows from the north and the vector representing opposition to this wind has the same magnitude, but opposite direction.

If θ represents the direction of $\vec{c}$ as in the figure



$$\tan\theta = \tfrac{65}{350} \doteq 0.1857$$
$$\therefore \theta \doteq 11° \qquad \text{(from tables)}$$

If $|\vec{c}|$ represents the magnitude of $\vec{c}$, then

$$|\vec{c}| = \sqrt{350^2 + 65^2}$$
$$= \sqrt{122\,500 + 4225} = \sqrt{126\,725}$$
$$\doteq 356$$

The pilot must set a course at 356 knots, 11° north of east.

90 electrical degrees

EXAMPLE 3. *In an electrical series circuit, the current is the same anywhere in the circuit. Hence, the current is used as a reference line. In the vector diagram (Figure 9-9) the current, I, of 5 A is drawn as a horizontal reference vector.* V_R *and* V_L *represent voltage losses of 100 V and 75 V respectively. The two voltage drops,* V_R *and* V_L*, are 90 electrical degrees out of phase with each other. If the line voltage, V, is the vector sum of* V_R *and* V_L*, calculate the magnitude and direction of V.*

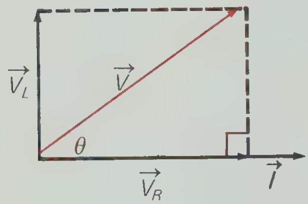


Figure 9-9

Solution

$$\vec{V} = \vec{V_L} + \vec{V_R}$$

$$|\vec{V}| = \sqrt{75^2 + 100^2}$$

$$= \sqrt{5625 + 10\,000}$$

$$= \sqrt{15\,625} = 125$$

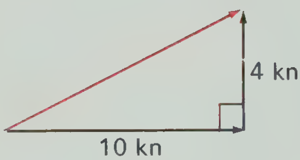
If θ is the angle between V and V_R, then

$$\cos\theta = \tfrac{100}{125} = 0.8000$$

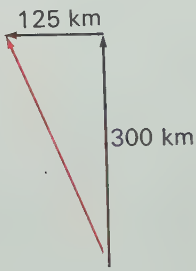
$$\theta = 37^\circ \qquad \text{(from tables)}$$

The line voltage is 125 V in a direction 37° counterclockwise to the current, and the current lags the voltage by 37°.

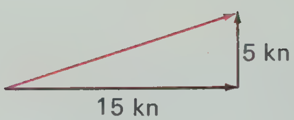
EXERCISE 9-8



B 1. A man attempts to row a boat at 10 knots directly across the 4 knot current of a river. What is his true speed and direction?

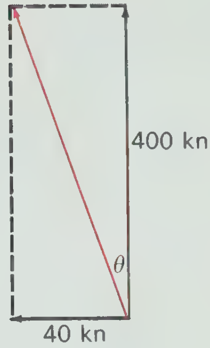


2. A ship sails 300 km north and 125 km west then develops engine trouble. Make a vector diagram then find the course a rescue ship would take to reach the stricken ship from the same port.

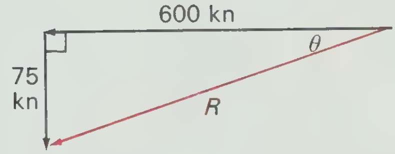


3. A small motorboat with a speed of 15 knots in still water is driven across a river that has a current of 5 knots. Find the actual speed and the true direction of the boat.

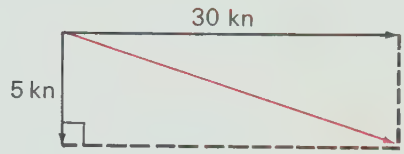
4. A plane is travelling north with an air speed of 400 knots. Find the ground speed and direction if there is a 40 knot wind from the east.



5. An airliner is heading east at 600 knots and encounters a 75 knot north wind. Find the resultant velocity and the true direction.

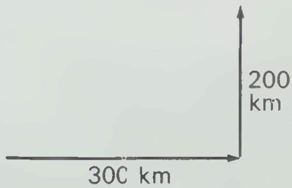


6. A pleasure craft is speeding across a river at 30 knots. Find the actual speed and direction if the current is 5 knots.

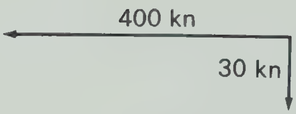


7. A pilot sets his course at 400 knots west relative to the ground. Find the true ground velocity and direction if the flight is affected by a 45 knot south wind.

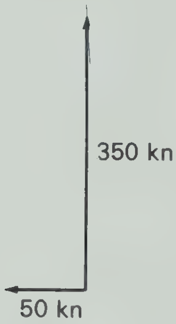
8. A plane flies 300 km east, then 200 km north. In what direction and how many kilometres should the aircraft fly to return to its home field?



9. In what direction should a pilot set her course if she wants to fly south at 300 knots and there is a 50 knot west wind?

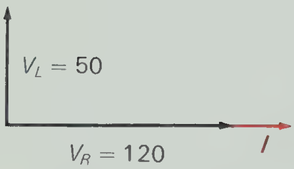


10. A pilot wishes to fly at 400 knots toward the west while there is a 30 knot south wind. Find the course he must set, and the air speed.

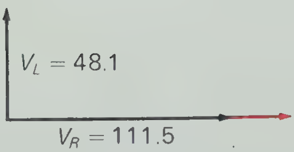


11. (a) What course must a pilot set to fly north if his air speed is 350 knots and there is a 50 knot west wind?
(b) Find the ground speed.

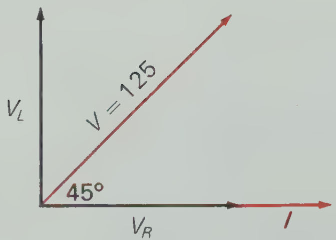
12. A pilot wishes to fly his aircraft directly east. What course must he set and what is his ground speed if there is a 50 knot north wind and the aircraft cruises at 300 knots?



13. Two voltage drops, $V_R = 120$ V and $V_L = 50$ V, are 90 electrical degrees out of phase with each other. Find the line voltage V, and its direction from the reference vector which represents a current I of 6 A.



14. Two voltage drops, $V_R = 111.5$ and $V_L = 48.1$, measured in volts, are 90 electrical degrees out of phase with each other. Find the line voltage V and its direction given that the horizontal reference vector represents a current, I, of 8 A.



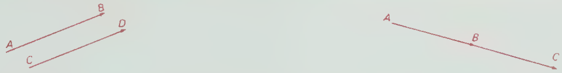
15. The line voltage, V, is given to be 125 V and the current, I, lags the voltage by 45°. Calculate the voltage losses, V_R and V_L, if it is known that they are 90 electrical degrees out of phase.

9.9 APPLICATIONS OF VECTORS TO GEOMETRY

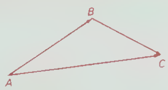
Vectors can be applied to geometrical problems. In this section, we shall make use of the basic properties of vectors studied in previous sections.

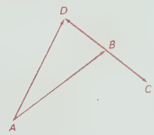
Some Geometric Properties of Vectors

1. $\overrightarrow{AB} = \overrightarrow{CD}$, if $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have the same magnitude and direction



2. If $\overrightarrow{AB} = \overrightarrow{BC}$, then $\overrightarrow{AC} = 2\overrightarrow{AB} = 2\overrightarrow{BC}$. $\overrightarrow{AB} = \overrightarrow{BC} = \frac{1}{2}\overrightarrow{AC}$
3. $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$





4. $\overrightarrow{AD} = \overrightarrow{AB} - \overrightarrow{BC}$, where $\overrightarrow{BD} = -\overrightarrow{BC}$

EXAMPLE 1. *M and N are the midpoints of AD and BC respectively in parallelogram ABCD. Show that AN is equal to and parallel to MC (i.e. show that $\overrightarrow{AN} = \overrightarrow{MC}$).*

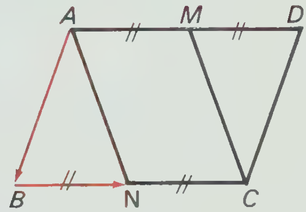


Figure 9-10

Solution Draw Figure 9-10 in your books and insert arrows so that the line segments represent vectors.

$$\begin{aligned}\overrightarrow{AN} &= \overrightarrow{AB} + \overrightarrow{BN}\\ &= \overrightarrow{DC} + \tfrac{1}{2}\overrightarrow{BC}\\ &= \overrightarrow{DC} + \tfrac{1}{2}\overrightarrow{AD}\\ &= \overrightarrow{DC} + \overrightarrow{MD}\\ &= \overrightarrow{MC}\end{aligned}$$

EXAMPLE 2. *Point M is the midpoint of the diagonal PR of parallelogram PQRS.*
(a) Prove that QM = MS
(b) Make a general statement concerning the diagonals of a parallelogram.

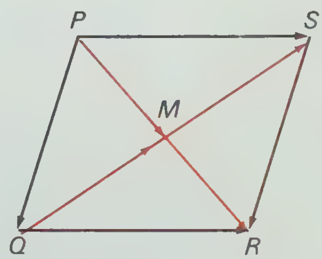


Figure 9-11

Solution Copy Figure 9-11 and mark arrows so that the line segments represent vectors.

(a) Given: Parallelogram *PQRS* with
$\overrightarrow{PS} = \overrightarrow{QR}$, $\overrightarrow{PQ} = \overrightarrow{SR}$, $\overrightarrow{PM} = \overrightarrow{MR} = \frac{1}{2}\overrightarrow{PR}$
Required: to prove $QM = MS$

Proof: $\overrightarrow{QM} = \overrightarrow{QP} + \overrightarrow{PM} = -\overrightarrow{PQ} + \overrightarrow{PM}$

$$= -\overrightarrow{SR} + \overrightarrow{MR} = \overrightarrow{MR} - \overrightarrow{SR}$$

$= \overrightarrow{MR} + \overrightarrow{RS}$

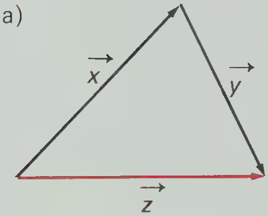
$= \overrightarrow{MS}$

(b) The diagonals of a parallelogram bisect each other.

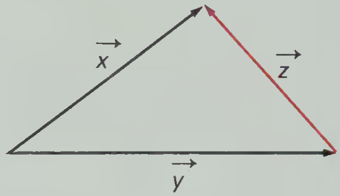
EXERCISE 9-9

A 1. Express $\vec{z}$ in terms of $\vec{x}$ and $\vec{y}$

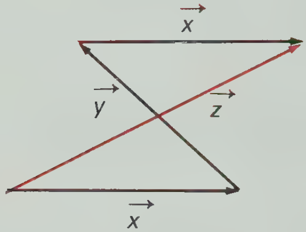
(a)



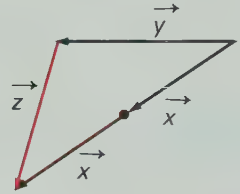
(b)

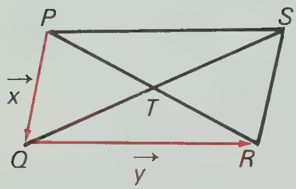


(c)



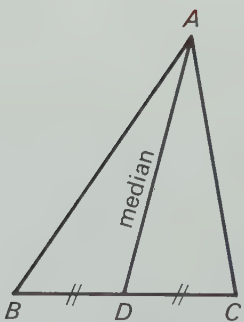
(d)





2. $PQRS$ is a parallelogram, where $\overrightarrow{PQ} = \vec{x}$ and $\overrightarrow{QR} = \vec{y}$. Express each of the following in terms of $\vec{x}$ and $\vec{y}$.

(a) $\overrightarrow{SR}$	(b) $\overrightarrow{PS}$	(c) $\overrightarrow{QP}$
(d) $\overrightarrow{PR}$	(e) $\overrightarrow{RS}$	(f) $\overrightarrow{QS}$
(g) $\overrightarrow{SQ}$	(h) $\overrightarrow{PT} + \overrightarrow{TR}$	(i) $\overrightarrow{PT}$
(j) $\overrightarrow{PQ} + \overrightarrow{QR}$	(k) $\overrightarrow{QT} + \overrightarrow{TS}$	(l) $\overrightarrow{QT}$
(m) $\overrightarrow{TS}$	(n) $\overrightarrow{ST}$	



B 3. D is the midpoint of BC in $\triangle ABC$. Make a vector diagram and find AD in terms of AB and BC.

4. *ABCD* is any quadrilateral with *P*, *Q*, *R*, and *S* the midpoints of *AB*, *BC*, *CD*, and *DA* respectively. Investigate Figure 9-12 and, using vectors:
(a) Find the relationship between *PQ* and *SR*.
(b) Identify *PQRS*.
(c) Check your answer to (b) using *PS* and *QR*.
(d) Identify the figure formed by joining the midpoints of a quadrilateral.

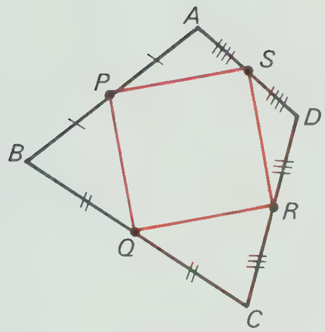


Figure 9-12

5. *M* and *N* are the midpoints of the sides *AB* and *AC* in $\triangle ABC$ as shown in Figure 9-13.
(a) Find the relationship between $\overrightarrow{MN}$ and $\overrightarrow{BC}$.
(b) How are *MN* and *BC* related?
(c) Make a general statement concerning the line joining the midpoints of two sides of a triangle and the third side.
(d) Check this statement using *P*, the midpoint of *BC*, and either *M* or *N*.

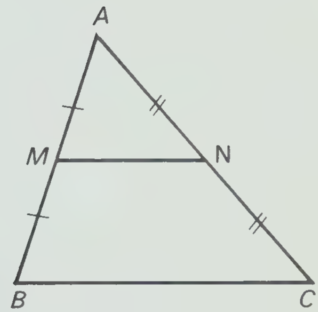


Figure 9-13

6. In $\triangle PQR$ (Figure 9-14), *S* and *T* are points on *PQ* and *PR* such that $ST = \frac{1}{2}QR$ and *ST* is parallel to *QR* (i.e. $\overrightarrow{ST} = \frac{1}{2}\overrightarrow{QR}$).
(a) Use vectors to show that $PS = SQ$.
(b) Show also that $PT = TR$.
(c) How does a line segment parallel to the base of a triangle divide the other two sides?

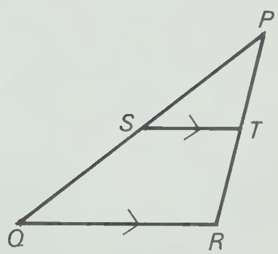


Figure 9-14

7. *X* is the midpoint of *PR* and *QS* in quadrilateral *PQRS* (Figure 9-15).
(a) Show that $PS = QR$ and that $PQ = SR$.
(b) Identify figure *PQRS*.
(c) Make a general statement concerning the above.

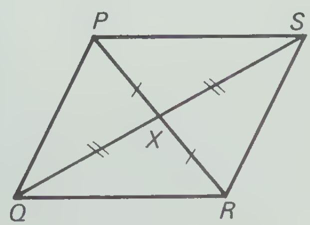


Figure 9-15

REVIEW EXERCISE

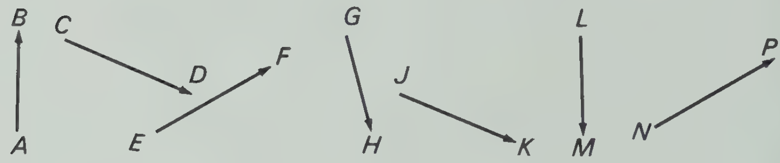
B **1.** Find the terminal point determined by the vector [4, −3] if the initial point is:

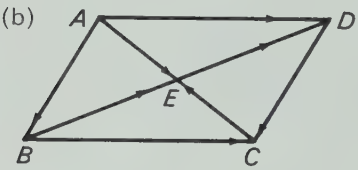
(a) (4, 0) (b) (−4, 3) (c) (0, 0)
(d) (4, −3) (e) (5, 1) (f) (x, y)

2. If $\vec{x} = [4, 2]$, $\vec{y} = [-1, 5]$, $\vec{z} = [4, -3]$, find:

(a) $3\vec{x}$ (b) $\vec{x} + \vec{z}$ (c) $4\vec{x} + 3\vec{y}$
(d) $\vec{x} - \vec{y}$ (e) $|\vec{x}|$ (f) $|\vec{x} + \vec{y}|$
(g) $|\vec{x}| + |\vec{y}|$ (h) $-(3\vec{x} + 2\vec{y} - \vec{z})$

3. Name pairs of equal vectors

(a)


(b)


4. Given the points 0(0, 0), $A(4, 2)$, $B(7, 5)$, $C(2, 0)$, $D(-6, 1)$.
(a) Draw a graph to show $\overrightarrow{OA}$, $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{OD}$.
(b) Express $\overrightarrow{OA}$, $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{OD}$, and $\overrightarrow{AC}$ in the form $[a, b]$.

5. Complete the following, given that $\vec{x} = [4, 7]$, $\vec{y} = [-2, 6]$, $\vec{z} = [0, -3]$:

(a) $\vec{x} + \vec{y} + \vec{z} =$ (b) $3\vec{x} + 2\vec{z} =$
(c) $\vec{x} + \quad = \vec{y} + \vec{z}$ (d) $\vec{x} - \quad = \vec{y}$
(e) $2\vec{y} = \quad + \vec{z}$ (f) $|\vec{x}| + |\vec{y}| =$
(g) $|\vec{x}| \times |\vec{y}| =$ (h) $|\vec{x} + \vec{y}| \quad |\vec{x}| + |\vec{y}|$

6. Given the vectors $\overrightarrow{OP} = [4, 2]$, $\overrightarrow{PQ} = [4, 2]$, $\overrightarrow{QR} = [7, 5]$, $\overrightarrow{RS} = [2, 0]$, $\overrightarrow{ST} = [-6, 1]$:
(a) Draw a graph to show the addition $\overrightarrow{OP} + \overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS} + \overrightarrow{ST}$.
(b) State the coordinates of P, Q, R, S, and T if the coordinates of 0 are:
(i) (0, 0) (ii) (−2, 4)

7. Given the points $A(4, 2)$, $B(7, 3)$, $C(-3, 0)$, $D(0, 1)$, determine which of $\overrightarrow{AB}$, $\overrightarrow{CD}$, $\overrightarrow{BA}$, and $\overrightarrow{DC}$ are equal vectors.

8. Find by means of a scale drawing the magnitude and direction of $\overrightarrow{AB} + \overrightarrow{BC}$, given that $\overrightarrow{AB}$ has a magnitude of 25 units with direction angle 30° and $\overrightarrow{BC}$ has a magnitude of 6 units with direction angle 312°.

9. Find the magnitude and direction of the resultant of the sum of the

following pair of vectors:

$\vec{x}$ has magnitude 5 units, south.

$\vec{y}$ has magnitude 12 units, west.

10. Given three points $A(-2, -2)$, $B(0, 3)$, $C(8, 5)$:

(a) Find the coordinates of point D by means of a diagram so that $\overrightarrow{AB} = \overrightarrow{DC}$.

(b) Express $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{AD}$, $\overrightarrow{DC}$, in the form $[a, b]$.

(c) Identify figure $ABCD$.

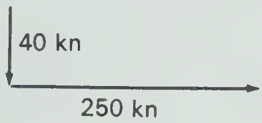
11. (a) Make a diagram to show $(-3)[4, -2] = [-12, 6]$

(b) Find x and y so that $[4, -2] = 2[x, y]$

12. State the magnitude and direction of each of the following vectors.

(a) $[4, 3]$ (b) $[3, 4]$ (c) $[-7, 11]$

(d) $[-5, -7]$ (e) $[15, -8]$ (f) $[a, b]$



13. An aircraft flies east at 250 knots air speed and encounters a north wind of 40 knots. Make vector diagrams and find:

(a) the ground speed and direction if the pilot does not adjust his course to account for the wind,

(b) the course the pilot should set to maintain his heading at 250 knots east.

14. In quadrilateral $ABCD$, $\overrightarrow{AB} = \overrightarrow{DC}$.

(a) Show that $\overrightarrow{AD} = \overrightarrow{BC}$ and make a general conclusion.

(b) Identify figure $ABCD$.

15. P and Q are the midpoints of AD and BC respectively in parallelogram $ABCD$.

(a) Make a vector diagram.

(b) Show that

(i) $ABQP$ (ii) $PQCD$ (iii) $AQCP$ (iv) $PBQD$

are parallelograms.

REVIEW AND PREVIEW TO CHAPTER 10

THE RULE OF THREE

In order to work with proportions we establish the Rule of Three, so called because it involves three steps.

EXAMPLE 1. *If a man earns \$38.50 for 7* h *of work, how much can he earn in 44* h*?*

Solution

$$\text{In 7 h he earns } \$38.50$$

$$\text{In 1 h he earns } \frac{\$38.50}{7}$$

$$\text{In 44 h he earns } 44 \times \frac{\$38.50}{7}$$

$$= \$242.00$$

EXERCISE

1. If 1.5 ℓ of soft drink cost 75¢, what is the cost of 20 ℓ?

2. If a 30 kg moon rock exerts a force of 294 N on the earth, what force is exerted by 18 kg of moon rocks on earth?

N = newtons.
A mass of 1 kg is attracted to the earth by a gravitational force of 9.8 N.

3. If 45 kg on the earth exert a force of 73.5 N on the moon, what force is exerted by an 85 kg man on the moon?

4. It requires 3805 J to vaporize 20 g of ice at 56°C. How many joules are required to vaporize 3 kg of dry ice?

5. One thousand transistors are required to manufacture 125 small radios. How many radios can be manufactured from a shipment of 469 120 transistors?

6. Twenty-two placemats can be cut from 2.75 m of material. How much material is required to cut 100 placemats?

7. A pendulum completes 22 vibrations in 34 s. How many vibrations will it complete in 50 s?

8. An engine will run for 3.25 h on 20 ℓ of fuel. What volume of fuel is required to run the engine for 24 h?

9. If 150 ml of toothpaste cost 99¢, what is the cost of 275 ml?

10. If 78.8 ℓ of sulphur dioxide can dissolve in 2 ℓ of water, how much will dissolve in 15 ℓ of water?

11. If 210 g of potassium nitrate will dissolve in 100 g of water to form a saturated solution at 20°C, how much will dissolve in 375 g of water at the same temperature?

12. If 8 g of oxygen occupy 5.6 ℓ at S.T.P., what volume will 50 g of oxygen occupy?

13. If 36.5 g of hydrogen chloride occupy 22.4 ℓ at S.T.P., what volume will 100 g occupy?

14. To form carbon monoxide, 12 g of carbon combine with 16 g of oxygen. How many grams of oxygen would have to combine with 100 g of carbon to form carbon monoxide?

Solve for the variable.

1. $a^2 = (41.32)^2 + (16.45)^2$

2. $a^2 = (1.676)^2 + (4.817)^2$

3. $x^2 = (0.7432)^2 + (0.8124)^2$

4. $y^2 = (157.3)^2 + (264.2)^2$

5. $(55.74)^2 = b^2 + (43.95)^2$

6. $(1.813)^2 = c^2 + (0.716)^2$

7. $(173.2)^2 = (95.1)^2 + y^2$

8. $(0.0813)^2 = (0.0546)^2 + x^2$

9. $m^2 = (5873)^2 + (4762)^2$

10. $x^2 = (55.55)^2 + (44.44)^2$

CHAPTER 10

Statics: Coplanar Forces

10.1 UNITS OF FORCE

Kinds of Forces
Chemical
Gravitational
Mechanical
Magnetic
Muscular
Nuclear
Electrical

Table 10-1

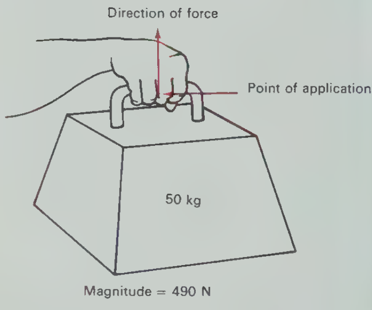
Forces are acting all around us. Our first experience with forces involves muscular exertion, such as throwing a ball, lifting a book, pushing a car, or pulling a wagon. Forces are also at work when muscular effort is absent—for example gravity and magnetism.

In figure 10-1 there are seven drawings of forces. Name the force illustrated in each drawing from the list in table 10-1.

In this chapter we shall concern ourselves with mechanical forces such as appear in the center drawing.

Three things must be known in order to describe a force:

F O R C E	1. Direction 2. Point of Application 3. Magnitude



Test your skill:

$$\frac{25 + 25}{25 \times 25}$$

The *units of force* are derived from Newton's Second Law of Motion.

$F = ma$, where F is the force
m is the mass
a is acceleration

The force which will impart to a mass of 1 kg an acceleration of 1 m/s has magnitude 1 kg.m/s² and is equal to 1 N (newton) of force.

$$\boxed{\begin{aligned} 1\ \text{N} &= 1\ \text{kg} \times 1\ \text{m/s}^2 \\ &= 1\ \text{kg}\cdot\text{m/s}^2 \end{aligned}}$$

EXAMPLE 1. *A force of 100* N *is applied to a mass of 20* kg. *What acceleration will be imparted to the 20* kg *body?*

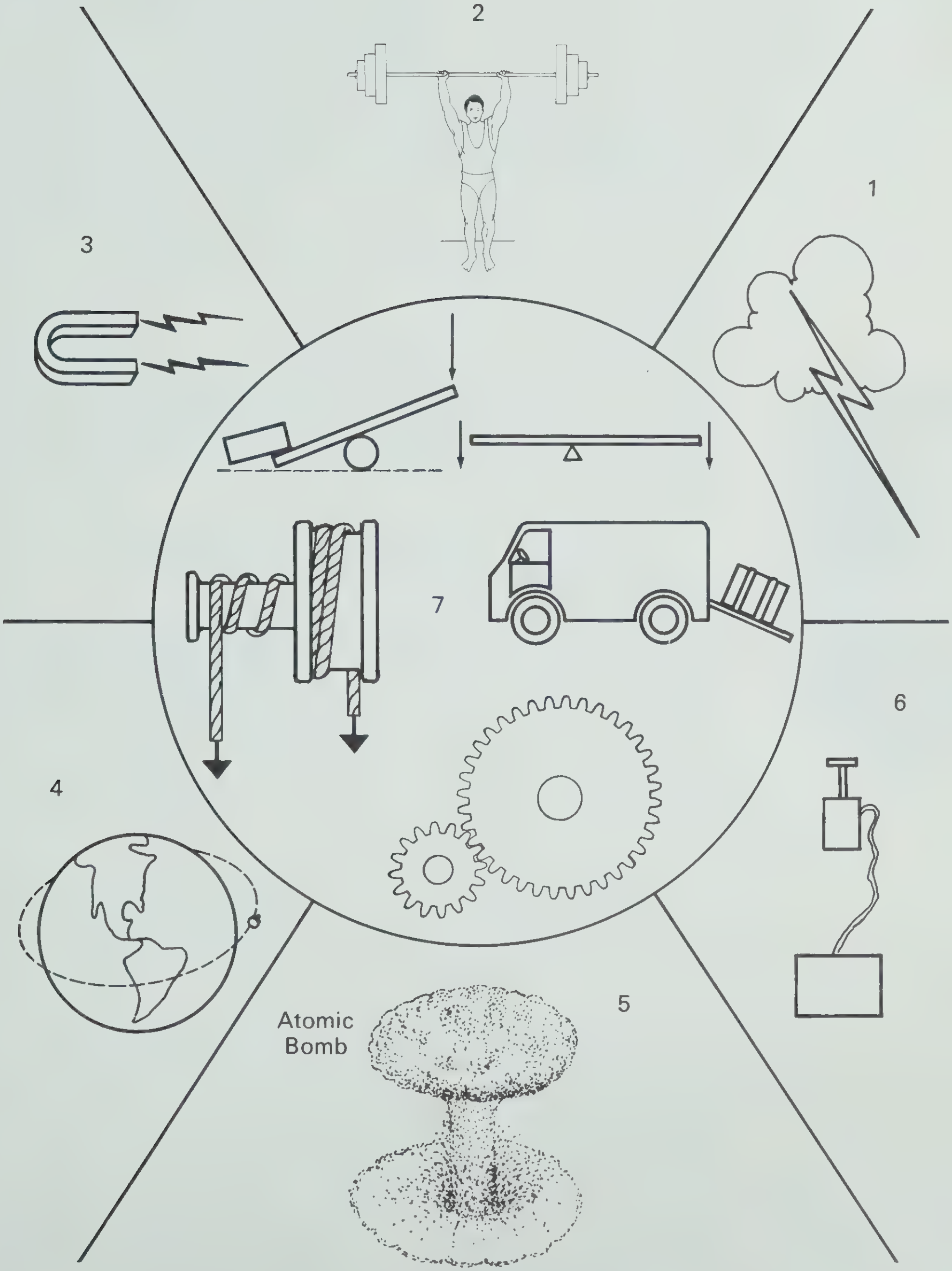


Figure 10-1

Solution

From Newton's second law

$$F = ma$$

$$100 \text{ N} = 20 \text{ kg} \times a \text{ m/s}^2$$

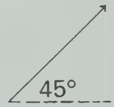
$$\frac{100}{20} \text{ m/s}^2 = a$$

$$5 \text{ m/s}^2 = a$$

The force produces an acceleration of 5 m/s².

EXAMPLE 2. *Using a scale of 1* N = *1* cm, *draw vectors to represent the following forces*
(a) 3 N *left* *(b) 2* N *down* *(c) 1* N *45° upwards to the right*

Solution
Scale 1 N = 1 cm
(a) 

(b) 2 N 
(c) 1 N 

Note that the scale is always stated when scale drawings are used.

Although the units of force have been derived using Newton's second law, this chapter is concerned with problems arising from the first and third laws.

Newton's First Law

When a body is at rest or moving with constant speed in a straight line, the resultant of all the forces exerted on the body is zero.

Newton's Third Law

Whenever one body exerts a force on another, the second always exerts on the first a force which is equal in magnitude but oppositely directed.

(This is sometimes stated: To every action there is an equal and opposite reaction.)


Sir Isaac Newton
Radio Times Hulton Picture Library

EXERCISE 10-1

B

1. Find the force that causes a 25 kg body to accelerate at 15 m/s².

2. Find the mass of a body that is accelerated at 20 m/s² by a force of 3 N.

3. Find the acceleration when a mass of 12 kg is acted upon by a force of 30 N.

4. Find the mass of a satellite hurtling through space with an acceleration of 75 m/s², caused by a force of 900 N.

5. Find the force of gravity if the earth attracts a mass of 1 kg toward itself, giving it an acceleration of 9.8 m/s².

6. Using a scale of 5 N = 1 cm make vector drawings of the following forces.
(a) 10 N to the right (b) 20 N to the left
(c) 15 N up (d) 25 N down
(e) 30 N to the left (f) 20 N up

7. Using a scale of 10 N = 5 cm make vector drawings of the following

forces.
(a) 30 N, 45° upwards and to the left.
(b) 40 N, 45° down and to the right.
(c) 35 N, 45° upwards and to the right.
(d) 20 N, 45° down and to the left.

Test your skill: $(0.03)^4$

10.2 THE RESULTANT

Collinear forces are forces that act in the same straight line. Consider two horizontal forces of 8 N and 6 N acting at a point P and both pulling to the right as in figure 10-2.

Scale: 2 N = 1 cm

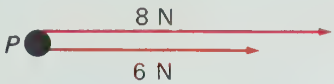


Figure 10-2

The *combined effect* of these two forces acting at P is the same as that of a single force of 14 N acting to the right as in figure 10-3.

Scale: 2 N = 1 cm

Figure 10-3

This single force that can take the place of the other two forces is called the resultant of the two forces.

The resultant of two noncollinear forces can be found geometrically by adding vectors as in chapter 9.

EXAMPLE 1. *Find the resultant of two forces of 6* N *and 8* N *acting at an angle of 60° to each other.*

Solution

Scale: 2 N = 1 cm

First we draw the vectors as in figure 10-4 and 10-5 using ruler, compasses, and protractor.

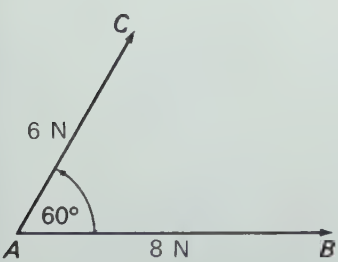


Figure 10-4

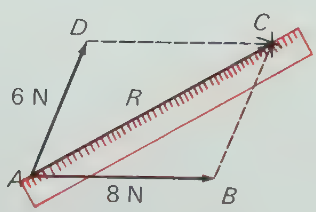
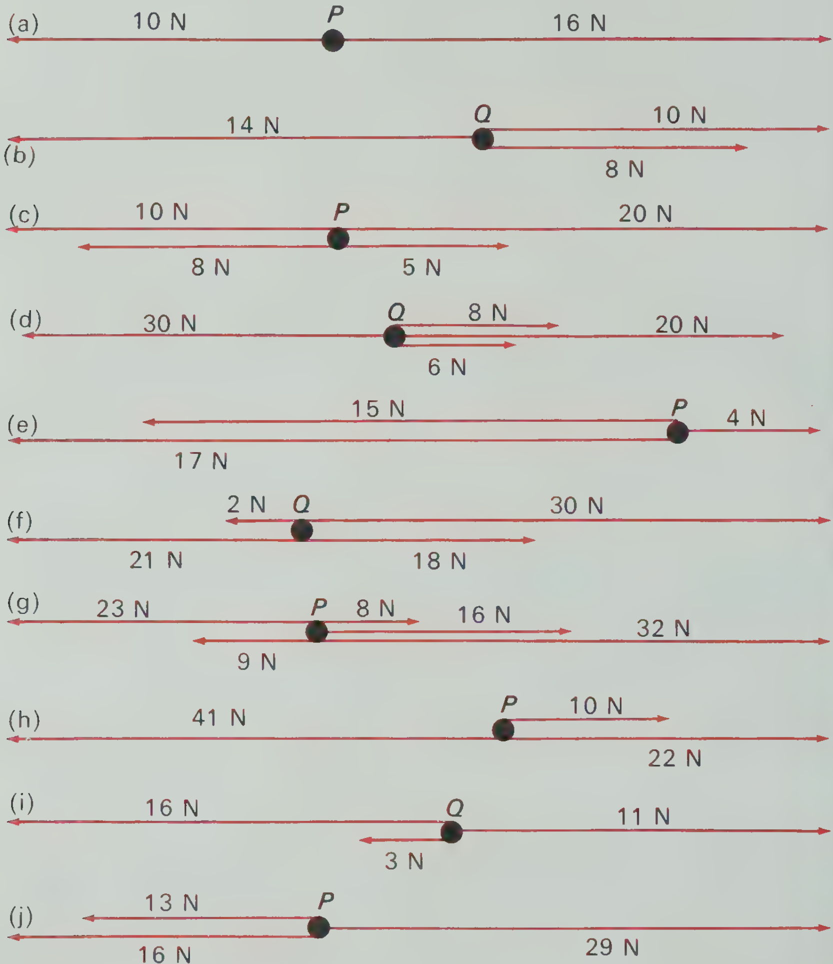


Figure 10-5

We then complete the parallelogram $ABCD$ and draw the diagonal AC. $\overrightarrow{AC}$ represents the resultant of the 6 N and 8 N forces. The length of $\overrightarrow{AC}$ is 6.1 cm so that the resultant is 12.2 N.

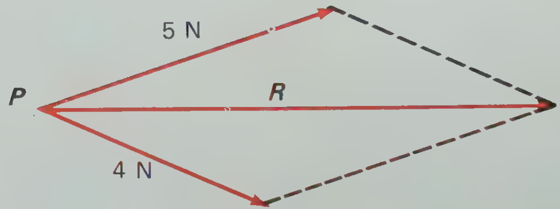
EXERCISE 10-2

A 1. Find the resultant of the following systems of forces.

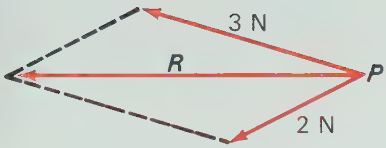


2. Using the scale of 1 N = 1 cm, determine (by measuring) the resultant in each of the following systems of forces.

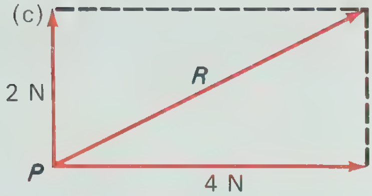
(a)



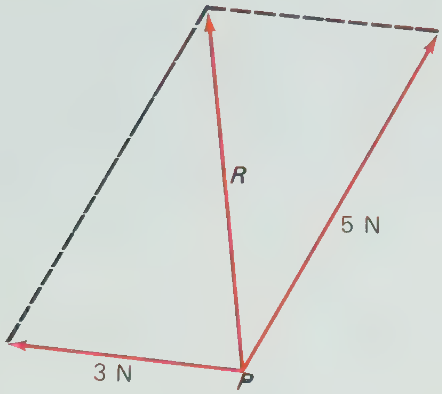
(b)



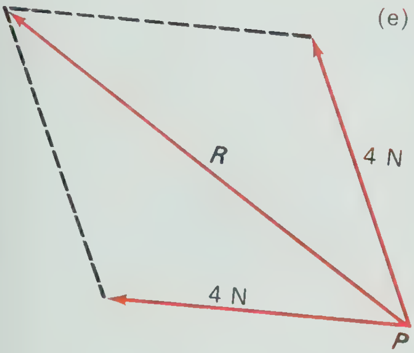
(c)



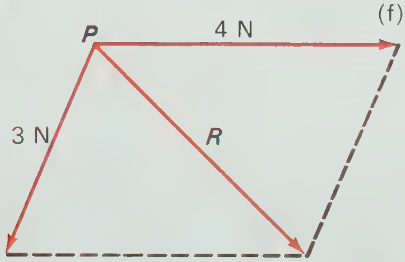
(d)



(e)

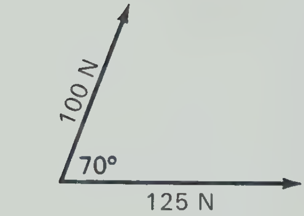


(f)



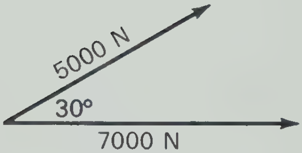
3. Using ruler, compasses, and protractor, find the resultant of the following systems of forces.

(a) 10 N and 15 N acting at 30° to each other.
(b) 8 N and 12 N acting at 70° to each other.
(c) 20 N and 30 N acting at 90° to each other.
(d) 6 N and 5 N acting at 80° to each other.
(e) 12 N and 16 N acting at 120° to each other.
(f) 9 N and 7 N acting at 130° to each other.
(g) 10 N and 20 N acting at 90° to each other.
(h) 7 N and 8 N acting at 45° to each other.

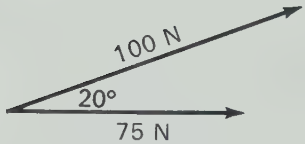


(i) 6 N and 5 N acting at 150° to each other.
(j) 11 N and 12 N acting at 80° to each other.

4. Two boys are pulling a log with forces of 100 N, and 125 N respectively. If the towing ropes make an angle of 70°
(a) make a scale drawing.
(b) find the magnitude of the resultant.
(c) measure the angle the resultant makes with the 125 N force.



5. Two tugs are towing a disabled ship. One tug exerts a force of 5 kN while the other exerts a force of 7 kN at an angle of 30° to the first tug :
(a) make a scale drawing.
(b) find the magnitude of the resultant.
(c) measure the angle the resultant makes with the 7 kN force.



6. Two boys are pulling a toboggan. One boy exerts a force of 75 N while the other boy exerts a force of 100 N. The ropes make an angle of 20° with each other:
(a) make a scale drawing.
(b) find the magnitude of the resultant.
(c) measure the angle the resultant makes with the 100 N force.

10.3 THE EQUILIBRANT

When a body is at rest or moving with constant speed in a straight line, it is said to be in equilibrium. If two forces of 6 N and 8 N, both acting to the right, act at a point, P, then the resultant is a force of 14 N acting to the right as in figure 10-6.

Scale 2 N = 1 cm

Multiply 142 857 by 2, 3, 4, 5, 6, and discover the pattern.

Figure 10-6

The equilibrant of this system of forces is the force that tends to prevent motion (i.e. produce equilibrium) and is equal to 14 N to the left as in figure 10-7.

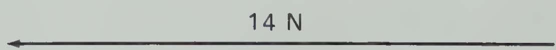


Figure 10-7

INVESTIGATION 10.3

1. (a) Set up three spring scales on a pegboard with golf tees (or similar arrangement) as shown in figure 10-8.
(b) Mark the readings on each scale in a table in your book. Change the positions of the golf tees so that you have four different sets of readings in your table.

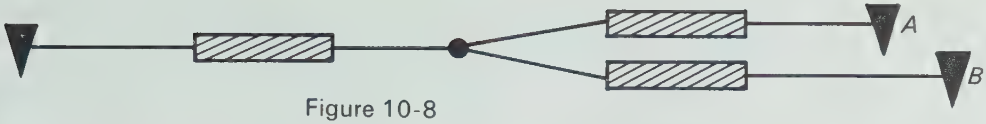


Figure 10-8

Force A	Force B	Resultant A + B	Equilibrant C

(c) How is the equilibrant related to the resultant with respect to
(i) magnitude? (ii) direction?

2. (a) Set up three spring scales on a pegboard with golf tees (or similar arrangement) as shown in figure 10-9.

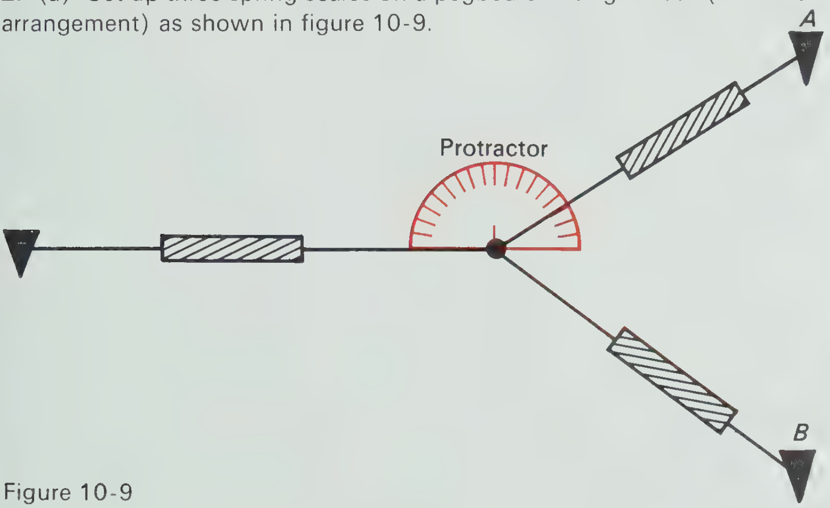
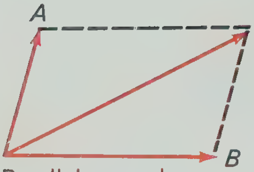


Figure 10-9

Using strings at the point of action of the forces makes it easier to read angles with a protractor.

(b) Mark the readings on each scale in a table in your book. Change the positions of the golf tees so that you have four different sets of readings in your table.

Force A	Force B	Equilibrant C	Resultant A + B



Parallelogram law

(c) Make scale drawings to find the resultant of the forces A and B for each of the four cases. Enter your result in your table.

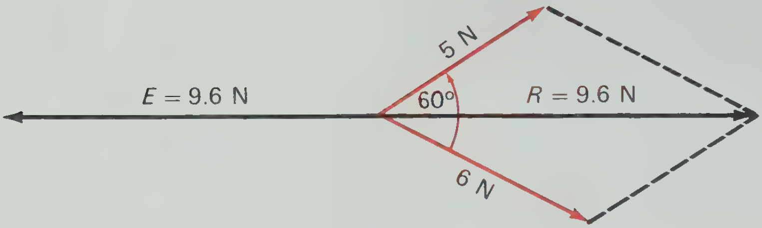
(d) How is the equilibrant, C, related to the resultant $A + B$ with respect to

(i) magnitude? (ii) direction?

EXAMPLE 1. *Find the equilibrant of 6* N *and 5* N *forces acting at an angle of 60° to each other.*

Solution

Scale: 2 N = 1 cm



1. Draw the vectors first.
2. Complete the parallelogram to get the resultant.
3. Draw the equilibrant equal in magnitude with opposite direction to the resultant.

EXERCISE 10-3

A

1. Determine the magnitude and direction of the resultant and equilibrant of the following systems of forces.

(a)

(b)

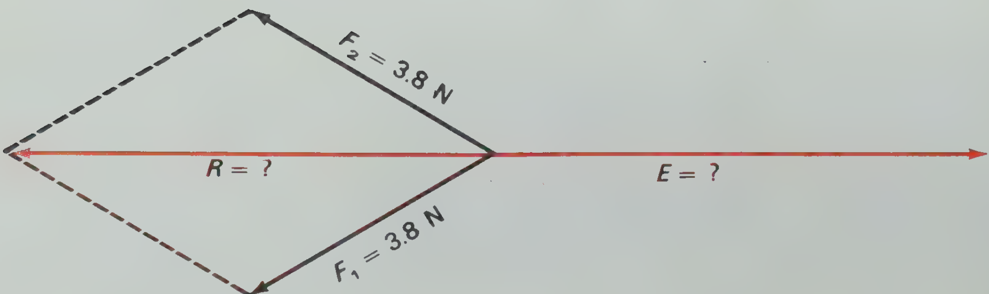
(c)

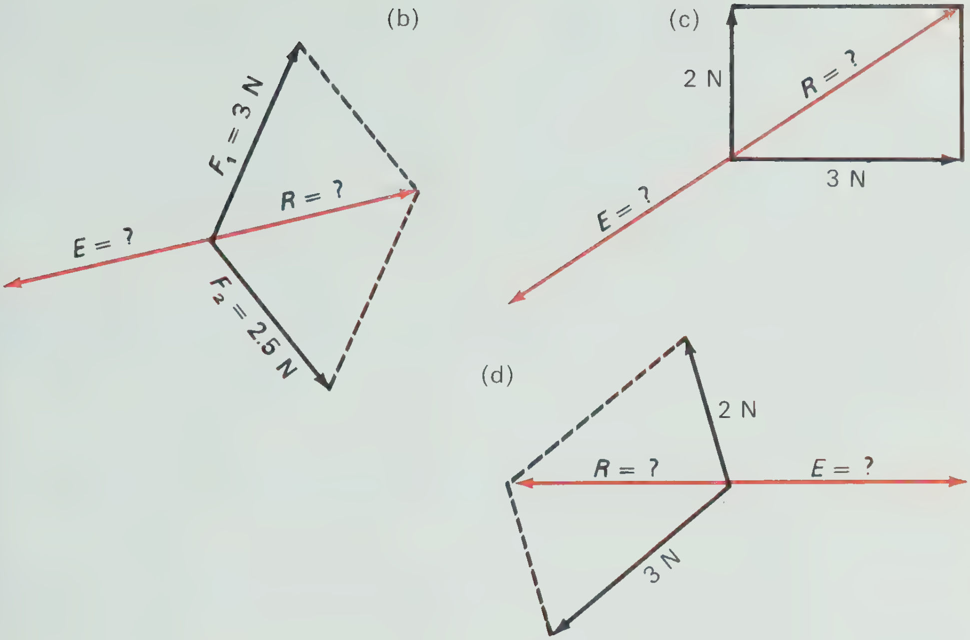
(d)

(e)

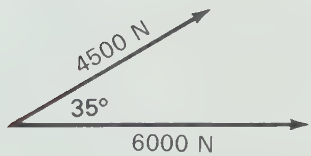
2. Using a scale of 1 N = 1 cm, determine by measurement the size of the resultant and equilibrant in each of the following systems of forces.

(a)

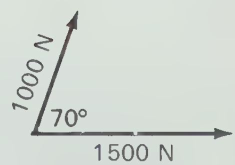




3. Using ruler, compasses, and protractor, find the resultant and equilibrant of the following systems of forces.
(a) 6 N and 8 N acting at 40° to each other.
(b) 10 N and 12 N acting at 70° to each other.
(c) 15 N and 20 N acting at 30° to each other.
(d) 30 N and 40 N acting at 90° to each other.
(e) 20 N and 25 N acting at 120° to each other.
(f) 8 N and 10 N acting at 150° to each other.
(g) 15 N and 10 N acting at 130° to each other.
(h) 12 N and 15 N acting at 90° to each other.

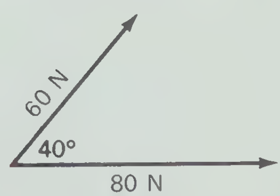


4. Two tugs tow a raft with forces 4.5 kN and 6 kN respectively. If the two ropes make an angle of 35° with each other, determine the equilibrant of the system.



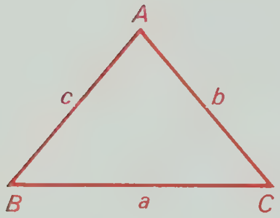
5. Two destroyers tow a mine with forces of 1 kN and 1.5 kN respectively. If the forces make an angle of 70° with each other, find the equilibrant.

6. Two boys pull a sled with forces of 80 N and 60 N respectively. If the angle between the forces is 40°, determine the magnitude of the equilibrant.



10.4 FINDING THE RESULTANT WITH TRIGONOMETRY

Trigonometry offers a more accurate way of finding the resultant of two noncollinear forces. Let two forces P and Q act at a point B making an angle θ. We complete the parallelogram of forces as in figure 10-10.

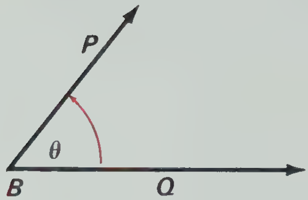


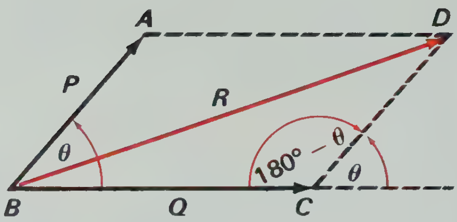
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Figure 10-10





BD is the resultant R of the two forces P and Q. In $\triangle BCD$, by the law of cosines,

$$R^2 = P^2 + Q^2 - 2PQ \cos (180° - \theta)$$

but $\cos (180° - \theta) = -\cos \theta$

$$R^2 = P^2 + Q^2 + 2PQ \cos \theta.$$

$$\boxed{R = \sqrt{P^2 + Q^2 + 2PQ \cos \theta}}$$

If $\theta = 90°$, or P and Q act at right angles to each other, then

$$\cos \theta = \cos 90° = 0$$

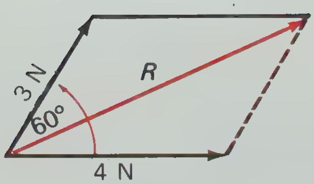
$$2PQ \cos \theta = 0$$

$$\boxed{R = \sqrt{P^2 + Q^2}}$$

The angle that R makes with one of the forces can be calculated using the law of sines.

EXAMPLE 1. *(a) Calculate the resultant of two forces of 3* N *and 4* N *acting at 60° to each other.*
(b) Calculate the angle that the resultant makes with the 4 N *force.*

Solution



(a) $R = \sqrt{P^2 + Q^2 + 2PQ \cos \theta}$

$$R = \sqrt{3^2 + 4^2 + 2 \times 3 \times 4 \times \cos 60°}$$

$$= \sqrt{9 + 16 + 24 \times 0.5}$$

$$= \sqrt{37}$$

$$\doteq 6.1$$

(b) To calculate the angle R makes with the 4 N force, we draw the parallelogram of forces. Let the angle between R and the 4 N force be α (alpha).

$$\angle B = 120°$$

In $\triangle ABC$, $\dfrac{\sin B}{R} = \dfrac{\sin \alpha}{3}$

$$\frac{\sin \alpha}{3} = \frac{\sin 120°}{6.1}$$

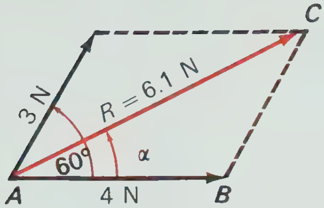
$$\sin \alpha = \frac{3 \times \sin 120°}{6.1}$$

$$= \frac{3 \times \sin 60°}{6.1}$$

$$\doteq \frac{3 \times 0.8660}{6.1}$$

$$\doteq 0.4259$$

$$\therefore \alpha \doteq 25°$$



$\sin 120°$
$= \sin (180° - 120°)$
$= \sin 60°$

The resultant is a 6.1 N force acting at an angle of 25° with the 4 N force.

EXAMPLE 2. *Calculate the resultant of two forces of 6 N and 8 N acting at an angle of 130° to each other.*

Solution

$$R = \sqrt{P^2 + Q^2 + 2PQ \cos \theta}$$

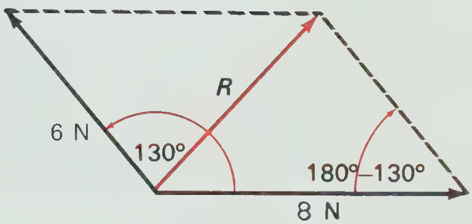
$$= \sqrt{8^2 + 6^2 + 2 \times 6 \times 8 \times \cos 130°}$$

$$\doteq \sqrt{64 + 36 + 2 \times 6 \times 8(-0.643)}$$

$$\doteq \sqrt{64 + 36 - 61.7}$$

$$\doteq \sqrt{38.3}$$

$$R \doteq 6.2$$



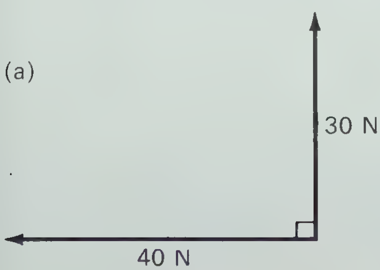
$\cos 130°$
$= -\cos (180° - 130°)$
$= -\cos 50°$
$= -0.6428$

The magnitude of the resultant is 6.2 N. The angle the resultant makes with one of the forces can be calculated using the law of sines.

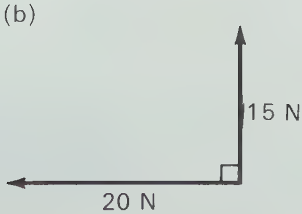
EXERCISE 10-4

B 1. Calculate the magnitude and direction (the angle made with the smaller force) of the resultant of the following systems of forces.

(a)

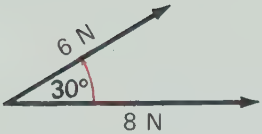


(b)

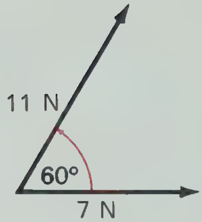


(c)

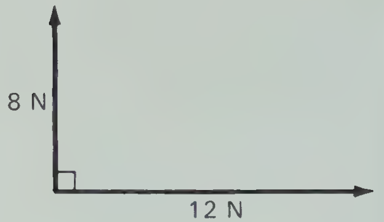
(d)



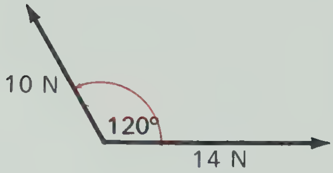
(e)



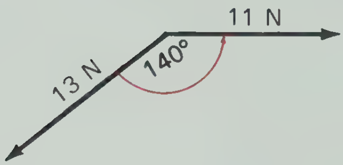
(f)



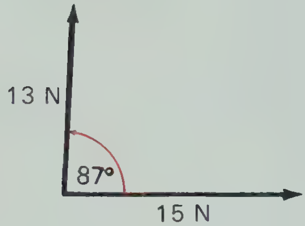
(g)

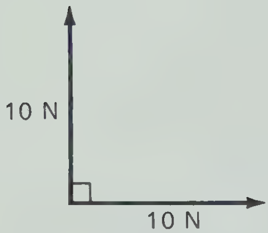


(h)

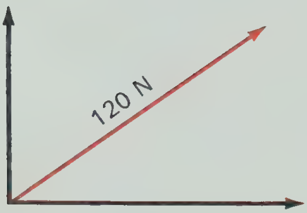


(i)

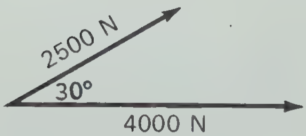




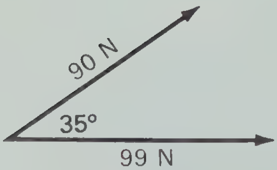
2. Two forces of 10 N each act on a body at 90° to each other. Find the magnitude and direction of the resultant.



3. Two boys with ropes at 90° to each other pull a sleigh with equal forces. Find the force applied to the sleigh if the tension in each rope is 120 N.

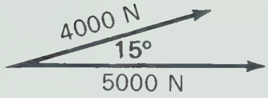


4. Two tow lines at 30° to each other have tensions of 2.5 kN and 4 kN. Find the magnitude of the resultant.

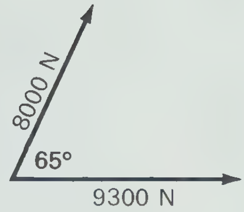


5. Two boys attempt to drag a log with forces of 90 N and 99 N respectively. If the angle between the two forces is 35°, what is the magnitude and direction of the force necessary to prevent the motion the boys tend to produce?

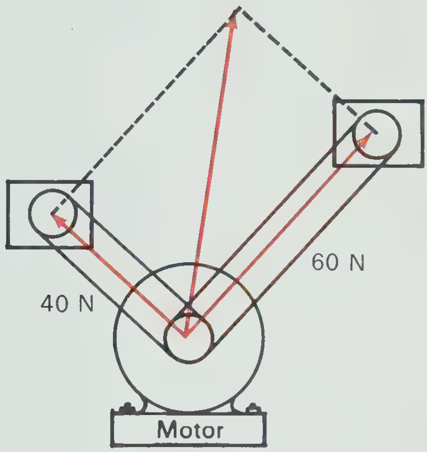
6. Two towtrucks pull a car with forces of 4 kN and 5 kN acting at 15° to each other.
(a) Find the magnitude of the resultant.
(b) In what direction will the car move with respect to the more powerful tow truck?



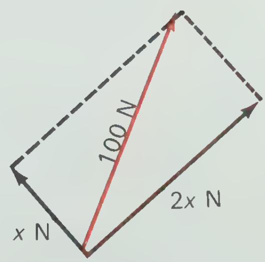
7. Two tugs tow a barge with forces of 8 kN and 9.3 kN, acting at 65° to each other.
(a) What is the resultant force on the barge?
(b) What angle does the resultant make with the 9.3 kN force?



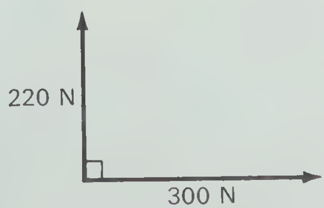
8. Two machines are driven by belts from the shaft of the same motor. The tensions in the belts cause forces of 40 N and 60 N acting at right angles to each other on the shaft of the motor as shown in the figure. Find the magnitude of the force on the shaft of the motor.



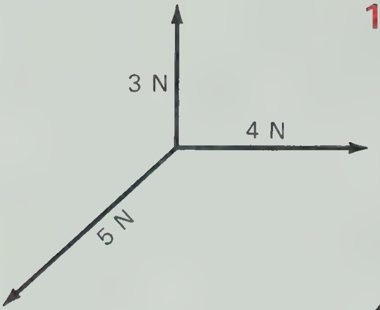
9. Two machines are driven from the shaft of the same motor by V-belts. The tension in one belt is double that in the other, and the belts are at right angles. Find the tension in each belt if the force on the motor shaft is 100 N.



10. Two boys pull on ropes attached to a load with forces of 300 N and 220 N at 90° to each other so that the load just moves. Find the magnitude and direction of the force exerted on the load.

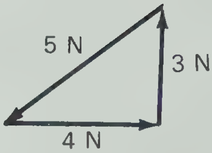


10.5 EQUILIBRIUM AT A POINT



Consider forces of 3, 4, and 5 N acting at a point. The system is in equilibrium if the forces are arranged so that the resultant is zero.

If a system of three forces in a plane is in equilibrium, the vectors representing the three forces can be arranged as the sides of a triangle.



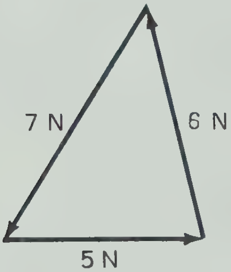


Figure 10-11

EXAMPLE 1. *Arrange forces of* 5 N, 6 N, *and* 7 N *so that they form a system in equilibrium at a point.*

Solution We first draw the triangle of forces as in figure 10-11.

Then we arrange the forces so that the vectors have a common initial point as in figure 10-12.

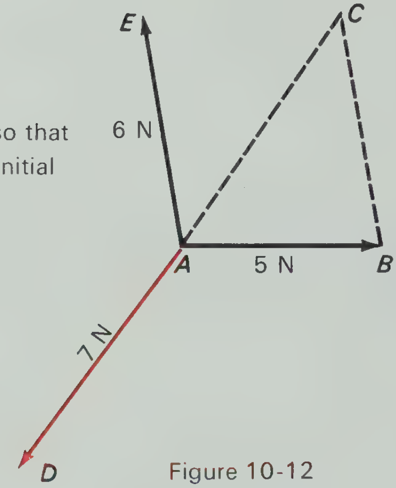


Figure 10-12

Since we are only concerned with magnitude and direction, vector $\overrightarrow{CA}$ can take position AD, vector $\overrightarrow{BC}$ can take position AE. The required directions of the forces can be determined from the geometry of the diagram.

The triangle is used to calculate the angles between the forces, using the law of cosines.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{7^2 + 5^2 - 6^2}{2 \times 7 \times 5}$$

$$= \frac{38}{70}$$

$$\doteq 0.5429$$

$$\angle A \doteq 57^\circ$$

$$\therefore \angle BAD = (180^\circ - 57^\circ) = 123^\circ$$

Similarly

$$\cos B = \frac{6^2 + 5^2 - 7^2}{2 \times 6 \times 5}$$

$$= \frac{12}{60}$$

$$= 0.200$$

$$\angle B \doteq 78°$$

$$\therefore \angle EAB = (180° - 78°) = 102°$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

To be in equilibrium the forces would be arranged so that the angle between the 5 and 7 N forces is 123°, and between the 6 and 5 N forces is 102°, as in figure 10-13.

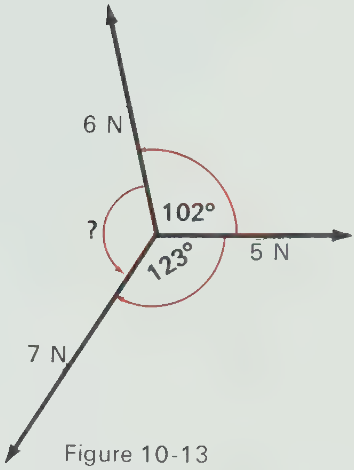


Figure 10-13

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

EXERCISE 10-5

A 1. State, with reasons, whether the following sets of forces can be arranged to form a system in equilibrium.

(a) 9 N, 10 N, 12 N
(b) 6 N, 10 N, 5 N
(c) 18 N, 20 N, 19 N
(d) 10 N, 15 N, 26 N
(e) 4 N, 4 N, 8 N
(f) 5 N, 6 N, 12 N

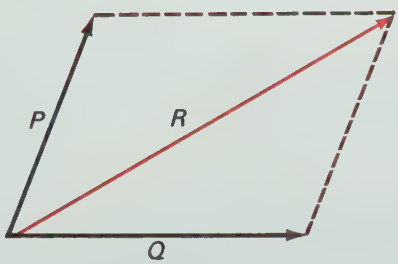
B 2. Arrange forces of 6 N, 7 N, and 8 N so that they act at a point to produce equilibrium. Find the angle between the forces by measurement and by calculation.

3. Three forces of 5 N, 12 N, and 13 N, act at a point to produce equilibrium. Calculate the angles between the forces.

4. Three forces of 6 N, 4 N, and 2 N act at a point to produce equilibrium. Calculate the angles between the forces.

5. Arrange forces of 1.1 kN, 1.2 kN, and 1.3 kN so they act at a point to produce equilibrium. Calculate the angles between the forces.

10.6 RESOLUTION OF FORCES



We have seen that the resultant of two forces is represented by the diagonal of the parallelogram of forces from the point of action. Conversely, we can take any force and express it as the diagonal of many parallelograms as in figure 10-14.

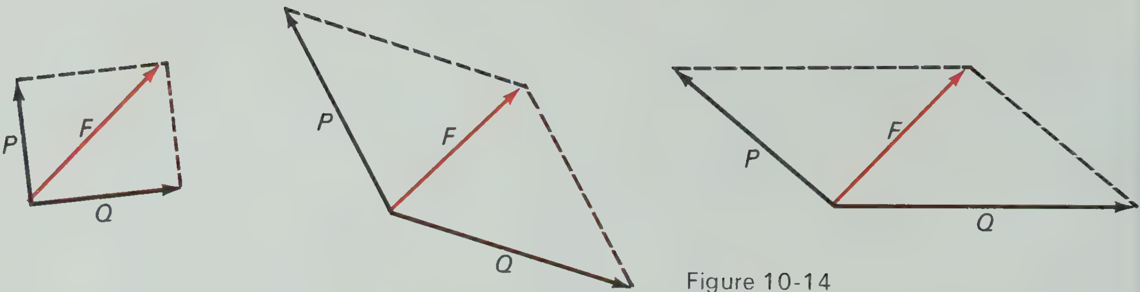


Figure 10-14

The forces P and Q are called the components of F. We say that the force F has been resolved into its components P and Q. There are many components of the same force F that are perpendicular to each other, as in figure 10-15.

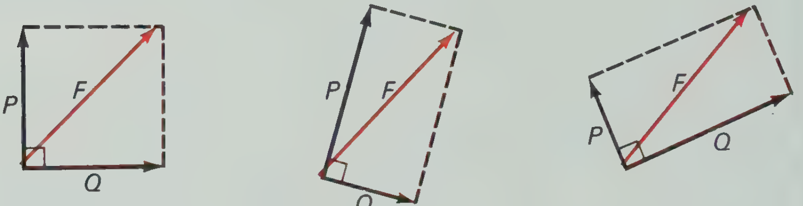
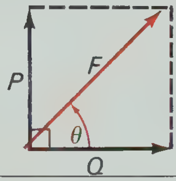


Figure 10-15

In this section we are interested in the components perpendicular to each other such that one is vertical and one is horizontal.

How many years are there in 100 centuries?



P is the vertical component of F
Q is the horizontal component of F
$P = F \sin \theta$
$Q = F \cos \theta$

EXAMPLE 1. *A player punts a football with a force of 150* N *at an angle of 40° to the horizontal (Figure 10-16).*
(a) What is the magnitude of the force that propels the ball forward?
(b) What force raises the ball?

Solution
(a) The horizontal component Q propels the ball forward.

Since $\dfrac{AB}{AC} = \cos 40°$

$$\frac{Q}{150} = \cos 40°$$

$$Q = 150 \times \cos 40°$$

$\doteq 150 \times 0.7660$

$\doteq 115$

Therefore a force of 115 N propels the ball forward.

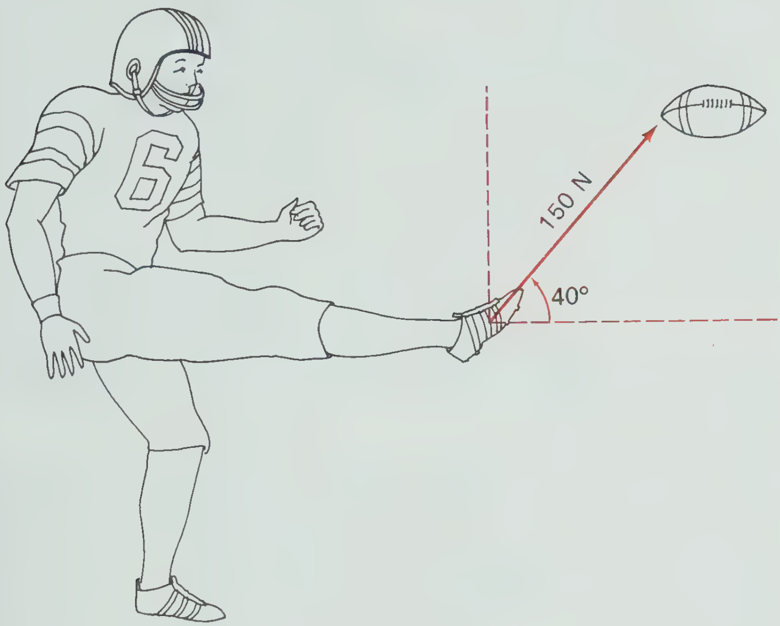
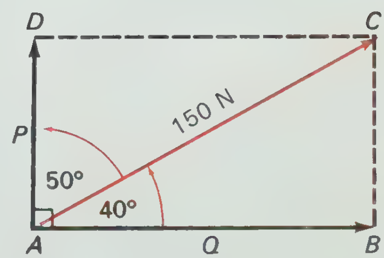


Figure 10-16

(b) The vertical component P raises the ball.



Since $\dfrac{AD}{AC} = \cos 50°$

$$\frac{P}{150} = \cos 50°$$

$$P = 150 \times \cos 50°$$

$$\doteq 150 \times 0.6428$$

$$\doteq 96$$

Hence, a force of 96 N propels the ball upward.

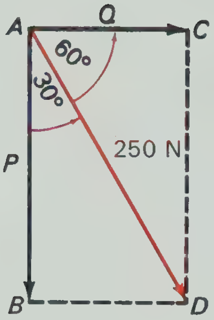
EXAMPLE 2. *When pushing a lawn mower, a man exerts a force of 250* N *along the handle.*

(a) If the handle is inclined at an angle of 60° to the ground, find the force that pushes the lawn mower forward.

(b) What is the magnitude of the force that tends to push the lawn mower into the ground?

Solution

(a) Since $\dfrac{AC}{AD} = \cos 60°$



$$\frac{Q}{250} = \cos 60°$$

$$Q = 250 \times \cos 60°$$
$$= 250 \times 0.500$$
$$= 125$$

A force of 125 N pushes the lawn mower forward.

(b) Since $\frac{AB}{AD} = \cos 30°$

$$\frac{P}{250} = \cos 30°$$

$$P = 250 \times \cos 30°$$
$$\doteq 250 \times 0.866$$
$$\doteq 217$$

A force of 217 N tends to push the lawn mower into the ground.

How many days are there in 100 centuries?

EXERCISE 10-6

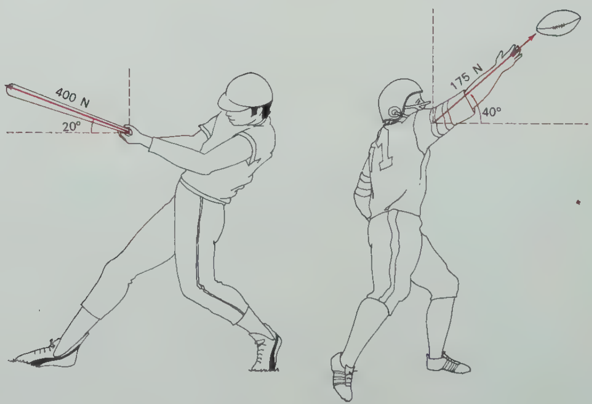
B 1. For each of the following forces, make a diagram and then calculate the magnitude of the vertical and horizontal components.

(a) 1000 N at 20° to the horizontal.
(b) 300 N at 60° to the vertical.
(c) 500 N at 10° to the horizontal.
(d) 600 N at 50° to the vertical.
(e) 120 N at 140° to the horizontal.

2. A boy pulls a sleigh exerting a force of 200 N along the rope which is at an angle of 35° to the horizontal. Find

(a) the force that moves the sleigh forward.
(b) the force that lifts the sleigh.

3. A man pushes a lawn mower with a force of 300 N directed down the handle. If the handle makes an angle of 50° with the ground, what is the magnitude of the force that tends to move the lawn mower forward? What is the magnitude of the force that pushes the lawn mower into the ground?

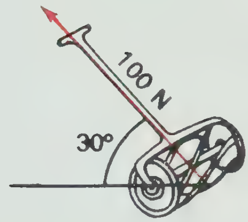


Henry Aaron hit record-breaking home run 715 at Atlanta on 1974-04-08.

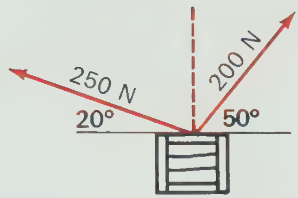
4. A batter hits a ball at an angle of 20° to the horizontal with a force of 400 N. Find the force that moves the ball forward and the force that raises the ball.

5. A quarterback throws a pass with a force of 175 N at an angle of 40° with the horizontal. Find the force that moves the ball forward and the force that raises the ball.

6. A boy pushes down the handle of a lawn mower with a force of 100 N. The handle makes a 30° angle with the ground.
(a) Find the magnitude of the force which acts to push the mower along the ground.
(b) Find the magnitude of the force which acts to push the mower into the ground.



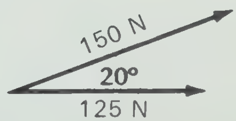
7. Two men try to raise a box with ropes. One man pulls with a force of 200 N at an angle of 50° with the horizontal. The other pulls with a force of 250 N at an angle of 20° with the horizontal.
(a) What is the total force that tends to raise the box?
(b) Will the box move toward the right or to the left?
(c) What is the force that tends to move the box to the right or left?



REVIEW EXERCISE

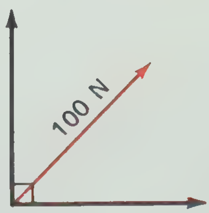
1. Use Newton's second law to find the force that imparts the given acceleration on the following bodies.
(a) 30 kg , 3.7 m/s^2 (b) 5 kg , 9.8 m/s^2
(c) 125 kg , 10 m/s^2 (d) 50 kg , 17 m/s^2

2. Find the magnitude of resultant of the following systems of forces.
(i) by scale drawing
(ii) using trigonometry
(a) 310 N and 250 N , at 90° to each other.
(b) 25 N and 40 N , at 90° to each other.
(c) 50 N and 50 N , at 90° to each other.
(d) 50 N and 50 N , at 60° to each other.
(e) 5 N and 10 N , at 30° to each other.
(f) 10 N and 17 N , at 45° to each other.

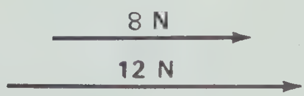


3. Two boys pull a toboggan with forces of 125 N and 150 N acting at 20° to each other. Find the resultant force on the toboggan.

4. Two equal forces act at 90° to produce a resultant of 100 N. Find the magnitude of the two equal forces.



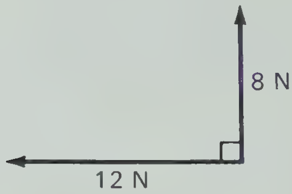
5. Find the equilibrant of two forces of 8 N and 12 N both acting to the right.



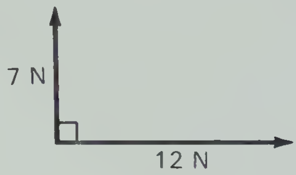
6. Use ruler, compasses, and protractor to draw the resultant and equilibrant of the following systems of forces.
(a) 10 N and 15 N, acting at 45° to each other.
(b) 350 N and 125 N, acting at 20° to each other.
(c) 300 N and 400 N, acting at 90° to each other.
(d) 50 N and 120 N, acting at 90° to each other.

7. Calculate the magnitude and direction (with respect to the larger force) of the resultant in each of the following.

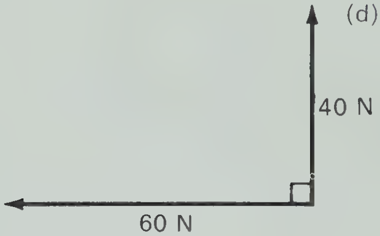
(a)



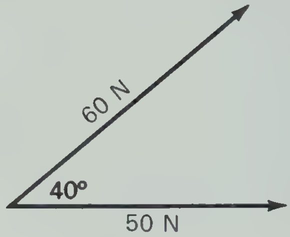
(b)



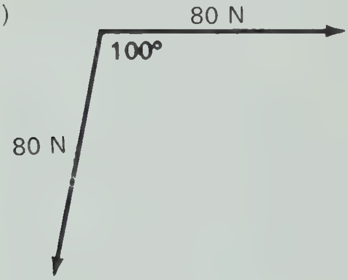
(c)



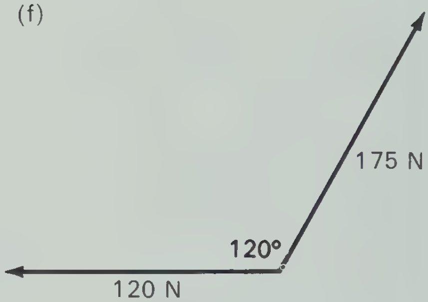
(d)

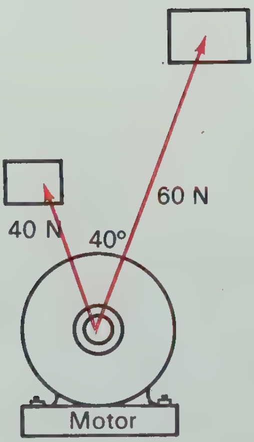


(e)



(f)

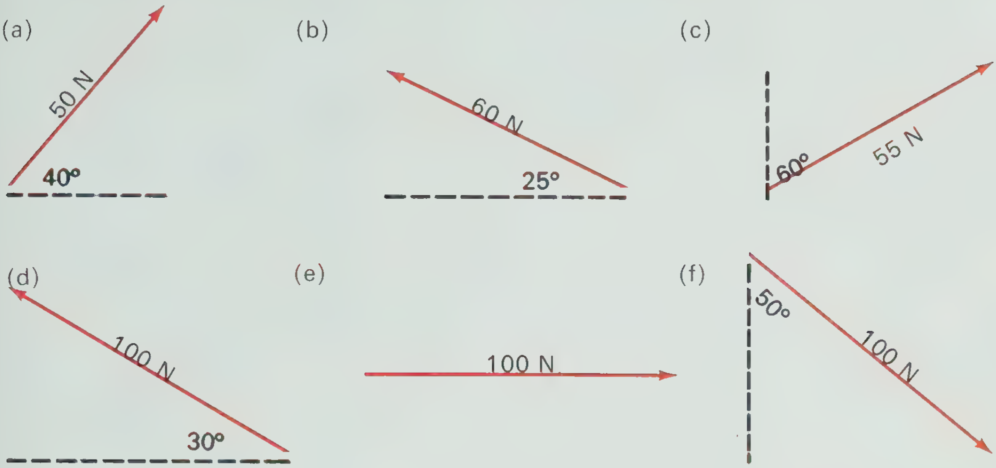




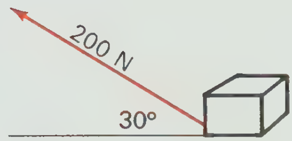
8. Two machines are driven from the shaft of the same motor. If the tensions in the belts are 40 N and 60 N, and the belts make an angle of 40° with each other, find the force on the shaft of the motor.

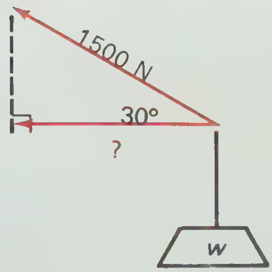
9. Arrange the following sets of forces to form systems in equilibrium.
(a) 15 N, 8 N, 17 N
(b) 8 N, 12 N, 15 N
(c) 12 N, 8 N, 4 N
(d) 120 N, 300 N, 190 N

10. For each of the following forces, find the vertical and horizontal components.



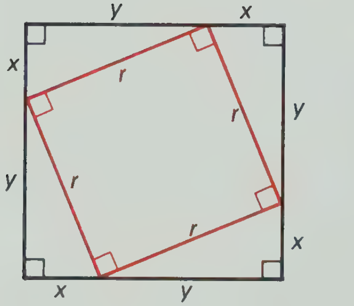
11. A box is pushed along the floor by a force of 200 N, making an angle of 30° with the horizontal. Find the horizontal and vertical components of the force.





12. A weight is suspended on the end of a cable which is held out by a horizontal boom. Find the force on the boom if the tension in the cable is 1.5 kN and the angle between the cable and the boom is 30°.

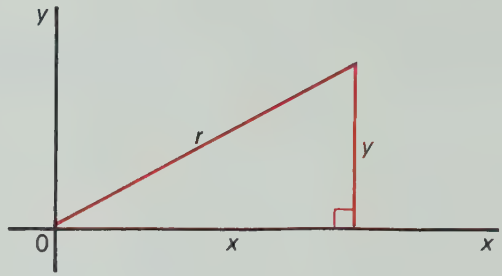
PYTHAGOREAN THEOREM



$$(x + y)^2 = r^2 + 4(\tfrac{1}{2}xy)$$

$$x^2 + 2xy + y^2 = r^2 + 2xy$$

$$x^2 + y^2 = r^2$$



$$r^2 = x^2 + y^2 \qquad r = \sqrt{x^2 + y^2}$$

$$x^2 = r^2 - y^2 \qquad x = \sqrt{r^2 - y^2}$$

$$y^2 = r^2 - x^2 \qquad y = \sqrt{r^2 - x^2}$$

EXAMPLE 1. *Solve for the variable.*

(a) $x = 4 \quad y = 3 \quad r = ▨$

$$r = \sqrt{4^2 + 3^2}$$
$$= \sqrt{16 + 9}$$
$$= \sqrt{25}$$
$$= 5$$

(b) $x = 5 \quad y = ▨ \quad r = 13$

$$y = \sqrt{13^2 - 5^2}$$
$$= \sqrt{169 - 25}$$
$$= \sqrt{144}$$
$$= 12$$

EXERCISE 1

1. Find the exact value for the missing variable (Figure 11-1).

Figure 11-1

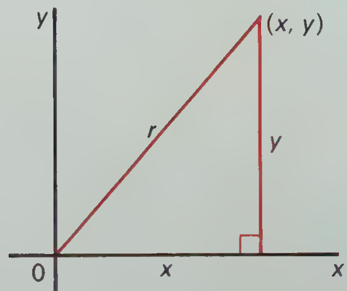
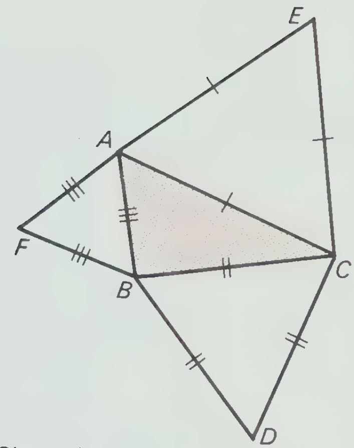


Figure 11-1

	r	x	y
a	10	8	
b		7	24
c	26	10	
d	13		12
e	$\sqrt{2}$	1	
f		$\sqrt{5}$	$2\sqrt{5}$
g	6	$\sqrt{11}$	
h	7		$\sqrt{13}$
i		3	5
j	8	7	

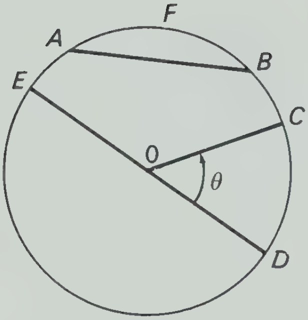


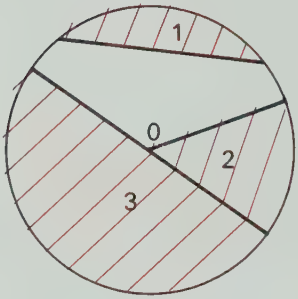
Show $\triangle EAC$
$= \triangle AFB + \triangle BDC$

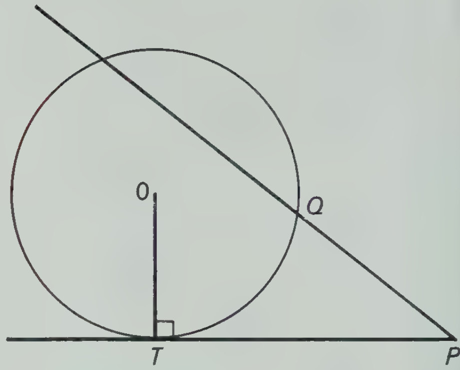
2. Use a slide rule to find the approximate value (to 2 significant figures) for the missing variable.

	r	x	y
a	7.0	4.0	
b	3.6		2.1
c		7.4	13
d	8.0	2.5	
e	14	10	
f		15	20
g	25	16	
h	15		10
i	2.5	0.7	
j	12	16	

Parts of a Circle



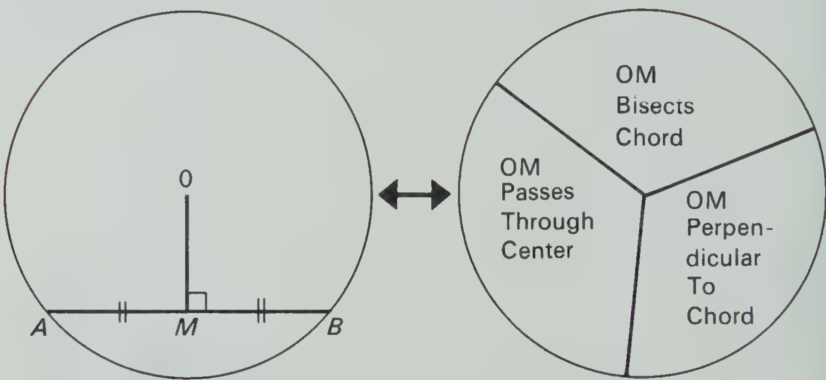




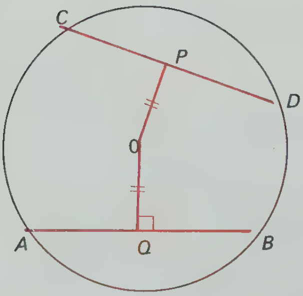
O Centre
AB Chord
OC Radius
EOD Diameter
AFB Arc
θ Central Angle

1 Segment (minor)
2 Sector (minor)
3 Semicircle

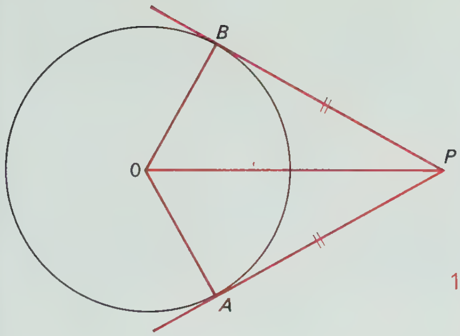
PQR Secant
PT Tangent
OT Radius



OM's Law
"Given two,
The third is true."

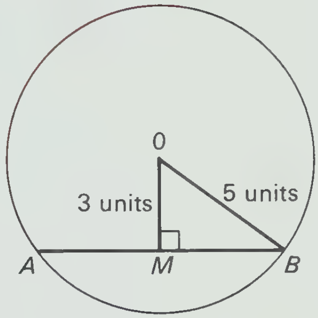


1. Chords equidistant from the centre are equal.
2. Equal chords are equidistant from the centre.



1. Two tangents PA and PB can be drawn to a circle from an exterior point P.
2. $PA = PB$
3. $PA \perp OA$, $PB \perp OB$.

EXAMPLE 2. *(a) Find the length of chord AB.*

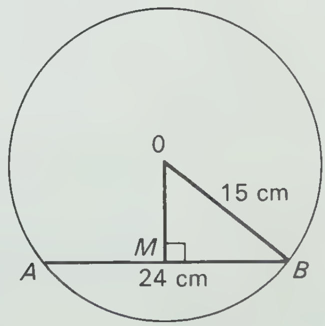


Solution

$$\begin{aligned} MB^2 &= 5^2 - 3^2 \\ &= 25 - 9 \\ &= 16 \\ MB &= 4 \text{ units} \\ AB &= 2MB \text{ (chord property)} \\ &= 2(4) \\ &= 8 \text{ units} \end{aligned}$$

AB is 8 units long.

(b) Find the distance from AB to the centre of the circle.

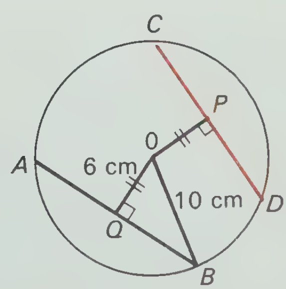


Solution

$$\begin{aligned} AB &= 24 \text{ cm} \\ MB &= 12 \text{ cm (chord property)} \\ OM^2 &= 15^2 - 12^2 \\ &= 225 - 144 \\ &= 81 \\ OM &= 9 \text{ cm} \end{aligned}$$

AB is 9 cm from the centre of the circle.

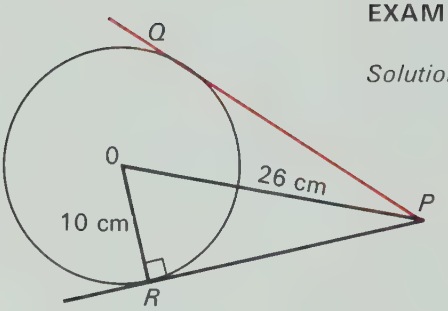
EXAMPLE 3. *Find the length of CD.*



Solution

$$\begin{aligned} QB^2 &= 10^2 - 6^2 \\ &= 100 - 36 \\ &= 64 \\ QB &= 8 \text{ cm} \\ AB &= 2QB \end{aligned}$$

$$= 16 \text{ cm}$$
$$CD = AB$$
$$= 16 \text{ cm}$$



EXAMPLE 4. *Find the length of tangent PQ.*

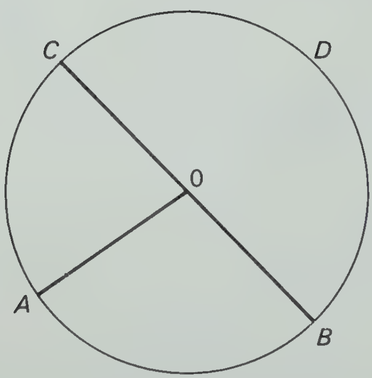
Solution

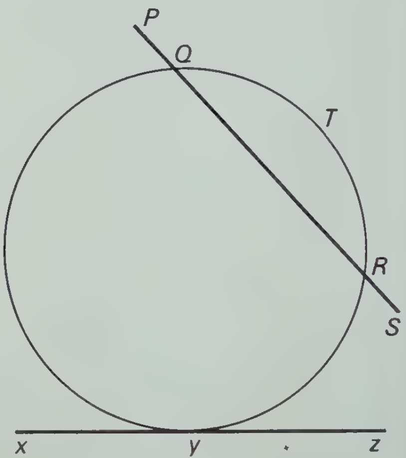
$$PR^2 = 26^2 - 10^2$$
$$= 676 - 100$$
$$= 576$$
$$PR = 24 \text{ cm}$$
$$PQ = PR$$
$$= 24 \text{ cm}$$

EXERCISE 2

1. From the diagrams name:

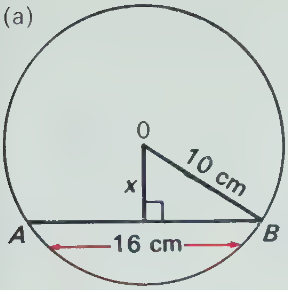
(a) a chord
(b) a tangent
(c) a diameter
(d) a segment (minor)
(e) a radius
(f) central angle
(g) a sector
(h) a segment (major)
(i) an arc
(j) a semicircle
(k) a secant



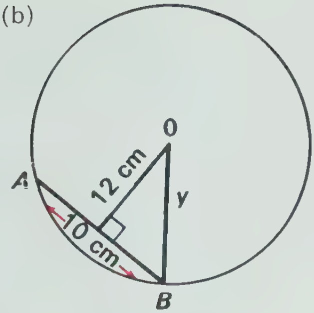


2. Evaluate the variables. O is the centre of the circle.

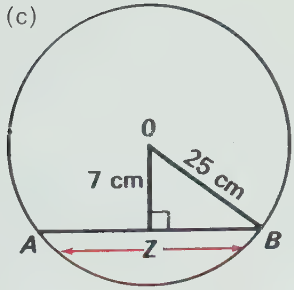
(a)



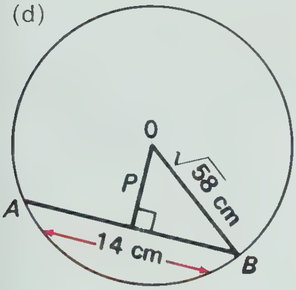
(b)



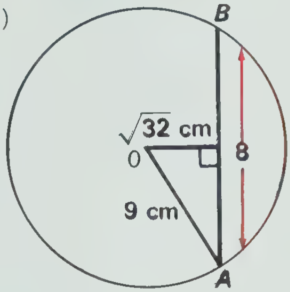
(c)



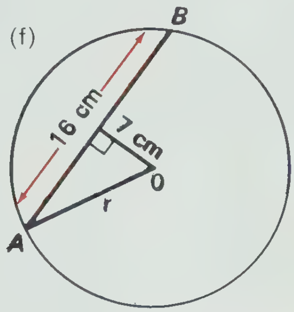
(d)



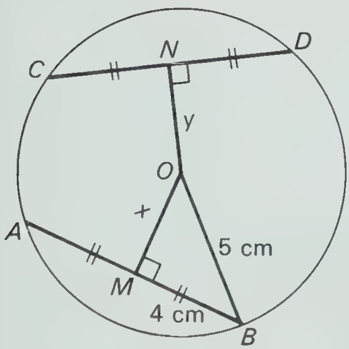
(e)



(f)

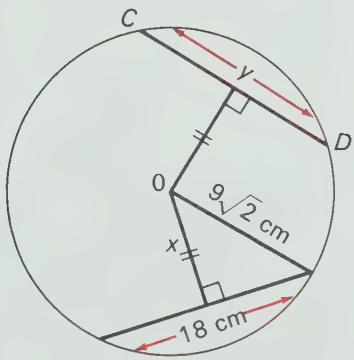


3. (a)



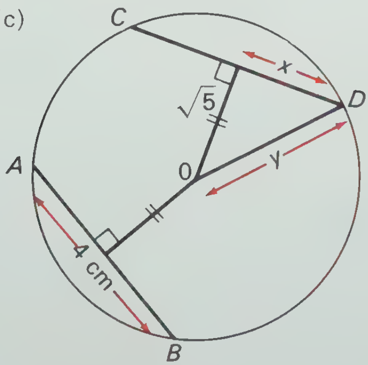
$x =$ ▨
$y =$ ▨

(b)



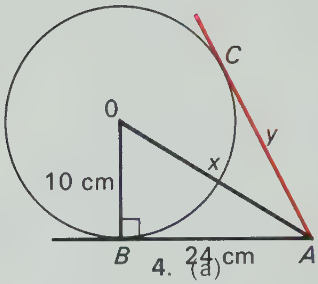
$x =$ ▨
$y =$ ▨

(c)



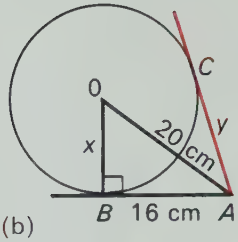
$x =$ ▨
$y =$ ▨

4. (a)



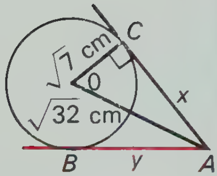
$x =$ ▨
$y =$ ▨

(b)



$x =$ ▨
$y =$ ▨

(c)



$x =$ ▨
$y =$ ▨

Equation Of A Straight Line

Given	Equation
Point (x_1, y_1), Slope m Points (x_1, y_1), (x_2, y_2) $\left[m = \dfrac{y_2 - y_1}{x_2 - x_1}\right]$	$y - y_1 = m(x - x_1)$
y intercept, b, slope m	$y = mx + b$

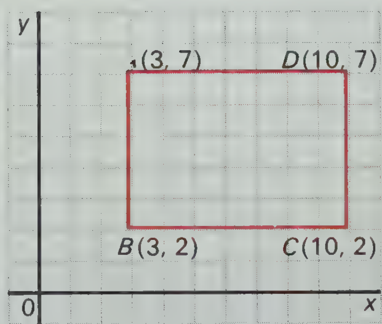
EXERCISE 3

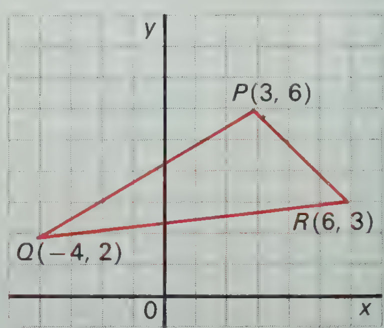
1. Find the equation of the lines given the following conditions.

(a) Point $(3, -2)$, slope $\frac{1}{2}$
(b) Point $(-1, 7)$, slope $-\frac{3}{5}$
(c) y intercept -4, slope -6
(d) y intercept $\frac{2}{3}$, slope 4
(e) Points $(-1, 5)$, $(9, 0)$
(f) Points $(7, 12)$, $(5, -2)$
(g) Point $(\frac{2}{3}, -\frac{1}{3})$, slope 2
(h) Point $(0, 6)$, slope $\frac{2}{3}$
(i) Points $(0, 6)$, $(3, 0)$
(j) Point $(2, 4)$, slope 0

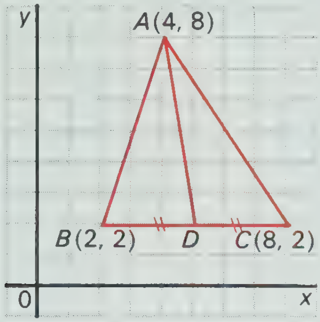
2. Find the equation of each side of the quadrilateral $ABCD$.

3. Find the equation of each side of the triangle PQR.

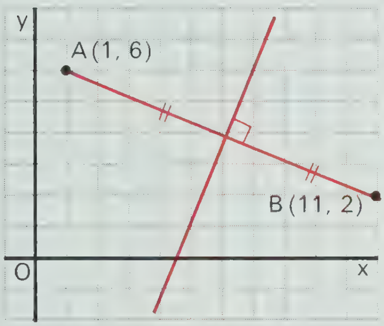


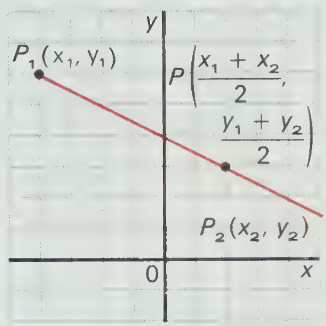


4. Find the equation of the median AD.



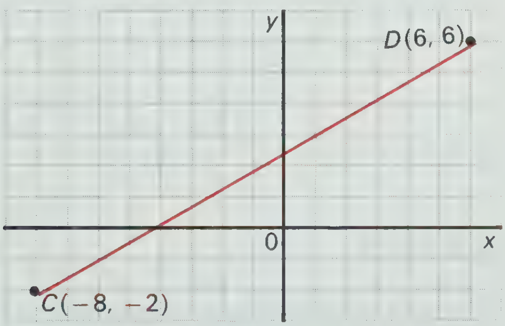
5. Find the equation of the right bisector of AB.





To find the midpoint of a line segment.

6. Find the equation of the right bisector of CD.

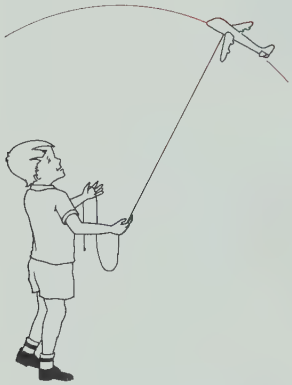


Solve for the variable (Use $\pi = 3.141\,59$).

1. $C = 2\pi(4.763)$
2. $C = 2\pi(0.4765)$
3. $C = 2\pi(156.7)$
4. $C = 2\pi(86.54)$
5. $C = 2\pi(587.4)$
6. $C = 2\pi(9764)$
7. $C = 2\pi(0.0763)$
8. $C = 2\pi(636.4)$
9. $C = 2\pi(79.86)$
10. $C = 2\pi(5.903)$

CHAPTER 11

The Circle

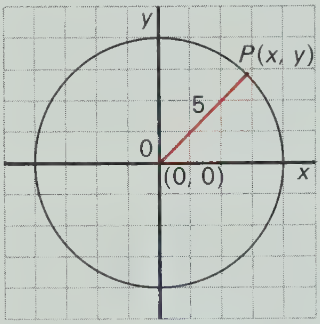


The early Greeks considered the circle to be the perfect curve. The wheel was one of man's first important inventions. The circular pistons in your car power the circular gears which turn the circular wheels. In this chapter we shall learn more about circles.

11.1 EQUATION OF A CIRCLE WITH CENTRE AT THE ORIGIN

> A circle is a set of points such that the length of the line segment joining any point of the set to a fixed point is constant.

(i) What is the fixed point called?
(ii) What is the constant length of the line segment called?



EXAMPLE 1. *Find the equation of the circle with centre at the origin and radius 5 units.*

Solution Any point $P(x, y)$ is on the circle if and only if the measure of OP is 5 units that is

$$\sqrt{(x-0)^2 + (y-0)^2} = 5$$

or $$\sqrt{x^2 + y^2} = 5$$

$\therefore$ $$x^2 + y^2 = 25 \quad \text{(squaring both sides)}$$

(i) Is the point (3, 4) on the circle?
(ii) Give the coordinates of two other points on the circle.

If and only if
—the point is on the curve
—the coordinates satisfy the equation.

EXERCISE 11-1

A

1. State the equation of the following circles.

(a) Centre (0, 0), radius 5
(b) Centre (0, 0), radius 9
(c) Centre (0, 0), radius 4
(d) Centre (0, 0), radius 12
(e) Centre (0, 0), radius 7
(f) Centre (0, 0), radius 8
(g) Centre (0, 0), radius $\sqrt{3}$
(h) Centre (0, 0), radius $\sqrt{2}$
(i) Centre (0, 0), radius a
(j) Centre (0, 0), radius r

The equation of a circle with centre O(0, 0) and radius r is

$$\boxed{x^2 + y^2 = r^2}$$

Unless otherwise stated, all variables represent real numbers.

2. State the centre and radius of the circle whose equation is given.

(a) $x^2 + y^2 = 16$ (b) $x^2 + y^2 = 100$
(c) $x^2 + y^2 = 25$ (d) $x^2 + y^2 = 5$
(e) $x^2 + y^2 = 36$ (f) $x^2 + y^2 = 49$
(g) $x^2 + y^2 = 12$ (h) $x^2 + y^2 = r^2$

B 3. (a) Find the radius of a circle whose centre is (0, 0) and which passes through the point (4, 0).
(b) Find the equation of a circle having centre (0, 0) and passing through (4, 0).

4. (a) What is the radius of a circle having centre (0, 0) and passing through the point (5, 12)?
(b) Find the equation of a circle having centre (0, 0) and passing through (5, 12).

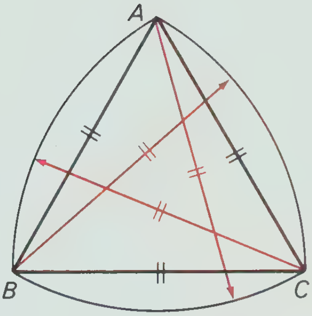
5. Find the equation of these circles.
(a) Centre (0, 0) passing through (3, 4)
(b) Centre (0, 0) passing through (24, −7)
(c) Centre (0, 0) passing through (3, 1)
(d) Centre (0, 0), y-intercept 7

6. Match each of the following equations of circles with the coordinates of the point which lies on it.

(a) $x^2 + y^2 = 10$ (i) (2, 5)
(b) $x^2 + y^2 = 32$ (ii) (1, −3)
(c) $x^2 + y^2 = 29$ (iii) (4, 0)
(d) $x^2 + y^2 = 16$ (iv) (−2, −3)
(e) $x^2 + y^2 = 13$ (v) (−4, −4)

7. Give the coordinates of two points that lie on each of the following circles.

(a) $x^2 + y^2 = 17$ (b) $x^2 + y^2 = 50$
(c) $x^2 + y^2 = 20$ (d) $x^2 + y^2 = 81$



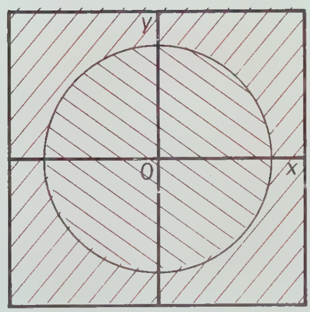
ABC is a curve of constant width. Would it work as a roller? What advantages would it have over a cylinder for this purpose?

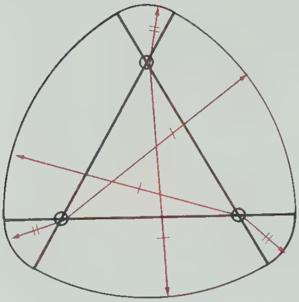
11.2 GRAPHS DETERMINED BY $x^2 + y^2 = r^2$

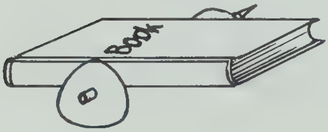
INVESTIGATION 11.2

1. (a) Graph the straight line whose equation is $y = 2x + 3$.
(b) Into how many regions does the line divide the plane?
(c) How would you describe each of these regions?

2. (a) Graph $x^2 + y^2 = 25$
(b) Into how many regions does $x^2 + y^2 = 25$ divide the plane?
(c) Where is the point (3, 4) with respect to the circle $x^2 + y^2 = 25$?
(d) Does (3, 4) satisfy $x^2 + y^2 = 25$?
(e) Where is the point (3, 7) with respect to the circle $x^2 + y^2 = 25$?
(f) Does (3, 7) satisfy $x^2 + y^2 = 25$?
(g) Where is the point (1, 1) with respect to the circle $x^2 + y^2 = 25$?
(h) Does (1, 1) satisfy $x^2 + y^2 = 25$?
(i) What is the inequality associated with the interior of the circle?
(j) What is the inequality associated with the exterior of the circle?







Construct two cardboard forms in the shape of a curve of constant width. Make a roller by putting one on each end of a pen or pencil. Try rolling it on a table under a book.

3. The equation $x^2 + y^2 = r^2$ represents all points on a circle with centre (0, 0), radius r. How would you represent all points interior to the circle? How would you represent all points exterior to the circle?

4. Determine whether each of the following points is a point of the interior of, the exterior of or on, the circle defined by $x^2 + y^2 = 25$.
(a) (0, 5) (b) (−3, −4) (c) (4.5, 1)
(d) (2, 3) (e) $(\sqrt{21}, 2)$ (f) $(\sqrt{24}, \sqrt{2})$
(g) $(\sqrt{17}, 2\sqrt{2})$ (h) (3.5, 4)

5. Sketch the graphs of the following inequalities.
(a) $x^2 + y^2 \leqq 1$ (b) $x^2 + y^2 > 4$
(c) $x^2 + y^2 < 25$ (d) $x^2 + y^2 \leqq 36$
(e) $x^2 + y^2 > 49$ (f) $x^2 + y^2 < 9$
(g) $x^2 + y^2 \geqq 100$ (h) $4x^2 + 4y^2 \leqq 49$
(i) $9x^2 + 9y^2 > 100$ (j) $4x^2 + 4y^2 < 81$

6. (a) Does the point (4, 3) lie on the circle $x^2 + y^2 = 25$?
(b) Does the point (4, −3) lie on the circle $x^2 + y^2 = 25$?
(c) These two points are said to be symmetric with respect to the x-axis. What are the coordinates of the point on the circle symmetric to (4, 3) with respect to the y-axis?
(d) The point (−4, −3) is symmetric to (4, 3) with respect to the origin. Is (−4, −3) on the circle? (It is because of this perfect symmetry that the Greeks considered the circle to be the perfect curve.)

7. (a) What is the y-intercept of $y = 2x + 3$?
(b) What are the y-intercepts of $x^2 + y^2 = 25$?
(c) How many y-intercepts does the circle have?
(d) What are the x-intercepts of $x^2 + y^2 = 25$?

8. State the x- and y-intercepts of each of the following circles.
(a) $x^2 + y^2 = 16$ (b) $x^2 + y^2 = 49$
(c) $x^2 + y^2 = 4$ (d) $x^2 + y^2 = 25$
(e) $x^2 + y^2 = 9$ (f) $x^2 + y^2 = 17$
(g) $x^2 + y^2 = 21$ (h) $x^2 + y^2 = 12$

11.3 THE EQUATION OF A CIRCLE WITH CENTRE NOT AT THE ORIGIN

In Section 11-1 we saw that the equation of a circle with centre (0, 0) and radius r is

$$x^2 + y^2 = r^2$$

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

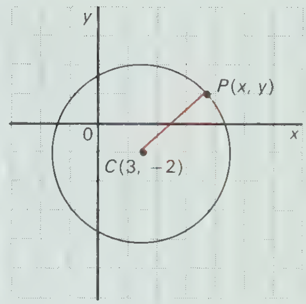
We shall now go on to look at cases where the centre is not at the origin.

EXERCISE 11-3

A

1. Calculate the following lengths AB.
(a) $A(0, 0)$ and $B(3, 1)$ (b) $A(1, 0)$ and $B(5, 3)$
(c) $A(2, 1)$ and $B(10, 7)$ (d) $A(-5, -2)$ and $B(-4, 7)$
(e) $A(1, 2)$ and $B(h, k)$ (f) $A(h, k)$ and $B(x, y)$

EXAMPLE 1. *Recall the definition of a circle and use it to find the equation of the circle having centre (3, −2) and radius 7.*



Solution Any point $P(x, y)$ is on the circle if and only if—The measure of CP is 7 units

that is— $\sqrt{(x-3)^2 + (y+2)^2} = 7$

or— $(x-3)^2 + (y+2)^2 = 49$ (squaring both sides)

B **2.** Write the equation of these circles.

(a) Centre (0, 3), radius 4 (b) Centre (1, 2), radius 8
(c) Centre (4, 0), radius 6 (d) Centre (3, 6), radius 7
(e) Centre (2, 4), radius 9 (f) Centre (5, 2), radius 12
(g) Centre (5, 2), radius $\sqrt{5}$ (h) Centre (1, 5), radius $\sqrt{6}$
(i) Centre $(\frac{1}{2}, \frac{1}{2})$, radius $\frac{3}{2}$ (j) Centre (h, k), radius r

The equation of a circle with centre (h, k) and radius r is

$$\boxed{(x-h)^2 + (y-k)^2 = r^2}$$

3. State the centre and radius of the circle whose equation is:

(a) $(x-4)^2 + (y-7)^2 = 121$ (b) $(x+5)^2 + (y-2)^2 = 25$
(c) $(x-2)^2 + (y-3)^2 = 36$ (d) $(x-2)^2 + (y+3)^2 = t^2$

4. Test this general formula by finding the equation of a circle with centre (0, 0) and radius r.

5. Find the equation of the following circles.

(a) Centre (4, 5), radius 3 (b) Centre (−1, 4), radius 5
(c) Centre (3, −4), radius 7 (d) Centre (6, 0), radius $3\sqrt{2}$

6. (a) Find the radius of the circle whose centre is (5, −3) and which passes through (1, 2).
(b) Find the equation of the circle in part (a).

7. Find the equation of the circle having its centre at the point of intersection of the lines represented by $x - y = 4$ and $x - 2y = 7$ and passing through the point (4, 3).

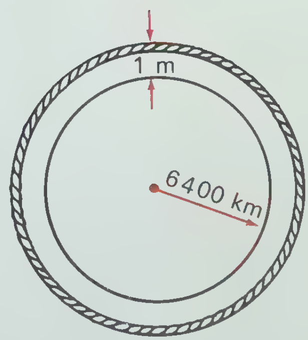
8. The equation of the circle found in Example 1 above was $(x-3)^2 + (y+2)^2 = 49$. This can be simplified by squaring the binomials and collecting like terms:

$$(x-3)^2 + (y+2)^2 = 49$$
$$\therefore x^2 - 6x + 9 + y^2 + 4y + 4 = 49$$
$$\therefore x^2 + y^2 - 6x + 4y - 36 = 0$$

Simplify the equations of the circles found in question 2.

9. Find the equation of the circle having a diameter with endpoints (5, −6) and (1, −4).

10. (a) Find the radius of a circle having centre (3, 2), and passing through (6, −2).
(b) State the equation of the circle: centre (3, 2), passing through (6, −2).



The equatorial *radius* of the earth is about 6400 km. How much *longer than the circumference* of the earth would a rope have to be in order to stretch around the earth 1 m above the surface?

(c) Suppose you were able to take this circle and move it 3 units to the left and 2 units down.

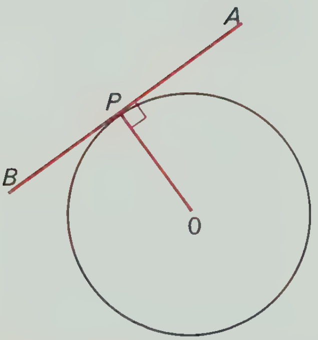
(i) What is the new centre?

(ii) What is the new radius?

(iii) What is the equation of this circle?

11. (a) State the equation of a circle with centre (0, 0) and radius 3.
(b) Translate this circle 4 units to the right and down 7 units. What are the centre and radius of the circle in this position?
(c) Using the values for the centre and radius found in (b) substitute into the equation of a circle with centre (h, k) and radius r.

11.4 EQUATION OF A TANGENT TO A CIRCLE



AB is a tangent to the circle at *P*.

P is the point of contact of the tangent and the circle.

A tangent to a circle is a line drawn perpendicular to a radius at the point of intersection of the radius and the circle.

EXAMPLE 1. *Find the equation of the tangent to the circle $x^2 + y^2 = 17$ at the point $P(-1, 4)$.*

Solution The tangent is a straight line; therefore, to find the equation we require a point on the tangent and the slope of the tangent.
Point $P(-1, 4)$ is on the tangent.
Slope $AB \perp OP$

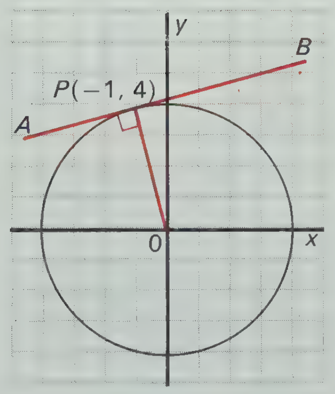
$$\therefore \text{ Slope } AB = -\frac{1}{\text{Slope } OP}$$

$$\text{Slope } OP = \frac{\triangle y}{\triangle x}$$

$$= \frac{0-4}{0-(-1)}$$

$$= -4$$

$$\therefore \text{ Slope } AB = \frac{1}{4}$$



$\therefore$ Equation of the tangent

$$y - y_1 = m(x - x_1)$$

$$y - 4 = \frac{1}{4}(x + 1)$$

$$4y - 16 = x + 1$$

$$x - 4y + 17 = 0$$

EXAMPLE 2. *Find the equation of the tangent to the circle with equation $(x-3)^2 + (y+4)^2 = 50$ at the point $P(4, 3)$.*

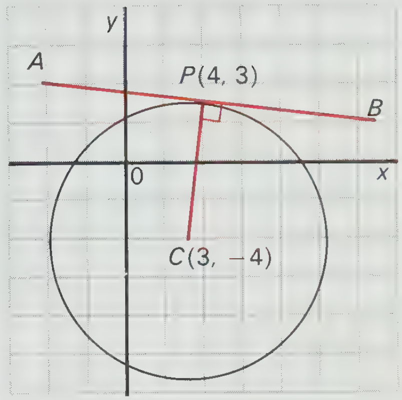
Solution Point $P(4, 3)$ is on the tangent.
Centre of the circle is $(3, -4)$.

$\therefore$ Slope of radius $CP = \dfrac{\triangle y}{\triangle x}$

$$= \frac{-4-3}{3-4}$$

$$= \frac{-7}{-1}$$

$$= 7$$



$\therefore$ Slope of the tangent $AB = -\frac{1}{7}$

$\therefore$ The equation of the tangent is

$$y - y_1 = m(x - x_1)$$

$$y - 3 = -\frac{1}{7}(x - 4)$$

$$-7y + 21 = x - 4$$

$$x + 7y - 25 = 0$$

EXERCISE 11-4

Parallel lines

$m_2 = m_1$ $\quad m_1 // m_2$

Perpendicular lines

$m_1 \times m_2 = -1$

$m_1 \quad m_2$

A 1. Find the slope of a line perpendicular to the following line segments.

(a) $A(-1, 3)$ $O(0, 0)$ (b) $B(4, 7)$ $O(0, 0)$
(c) $C(-2, 5)$ $O(0, 0)$ (d) $D(5, 0)$ $O(0, 0)$
(e) $E(4, 0)$ $O(0, 0)$ (f) $F(2, 3)$ $G(4, 7)$
(g) $H(-5, 7)$ $I(3, 9)$ (h) $J(7, 12)$ $K(-2, 15)$
(i) $L(2, -3)$ $M(5, -3)$ (j) $N(7, 2)$ $O(5, 2)$
(k) $P(6, 5)$ $Q(6, 7)$ (l) $R(-3, 0)$ $S(-3, 5)$
(m) $T(1, -2)$ $U(-3, 4)$ (n) $V(5, 1)$ $W(4, 0)$

2. Give in unsimplified form the equations of the following lines.

	Through point	Slope		Through point	Slope
(a)	(3, 4)	$\frac{1}{5}$	(b)	(1, −2)	2
(c)	(−4, 0)	$-\frac{2}{3}$	(d)	(5, 4)	$\frac{4}{3}$
(e)	(5, −2)	6	(f)	(6, −3)	undefined
(g)	(1, 7)	0	(h)	(−2, −1)	−4
(i)	(−1, 3)	∥ y-axis	(j)	(1, 0)	⊥ y-axis
(k)	(0, 6)	∥ x-axis	(l)	(3, −2)	⊥ x-axis

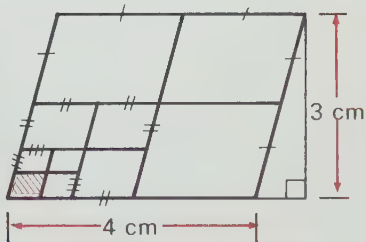
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

B 3. Find the equations of the tangents to the following circles at the given point of contact.

(a) $x^2 + y^2 = 25$ at $(3, 4)$ (b) $x^2 + y^2 = 10$ at $(-1, 3)$
(c) $x^2 + y^2 = 9$ at $(3, 0)$ (d) $x^2 + y^2 = 52$ at $(6, -4)$
(e) $x^2 + y^2 = 49$ at $(0, -7)$ (f) $x^2 + y^2 = 5$ at $(2, 1)$

4. Find the equations of the tangents to the following circles at the given point of contact.

(a) $(x - 4)^2 + (y + 3)^2 = 20$ at $(2, 1)$
(b) $(x + 1)^2 + (y - 2)^2 = 52$ at $(3, -4)$
(c) $(x - 2)^2 + (y + 1)^2 = 13$ at $(0, 2)$
(d) $x^2 + (y - 3)^2 = 16$ at $(4, 3)$



Find the area of the shaded rhombus.

(e) $(x - 5)^2 + y^2 = 8$ at $(7, 2)$
(f) $(x + 2)^2 + (y + 1)^2 = 36$ at $(-2, 5)$

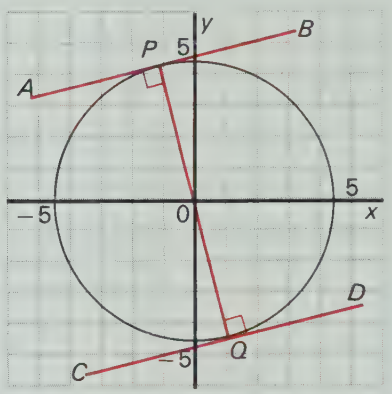
5. (a) Find y if $(2, y)$ defines a point on a circle with the equation $x^2 + y^2 = 13$.
(b) Illustrate the two solutions on a graph.
(c) Find the equations of the tangents to the circle with equation $x^2 + y^2 = 13$ at points of contact with x coordinate 2.

6. (a) Find the equations of the tangents to a circle with the equation $x^2 + y^2 = 26$ at points of contact with y coordinate -5.
(b) Find the equations of the tangents to the circle having equation $x^2 + y^2 = 16$ at point of contact.
(i) with x coordinate 0
(ii) with y coordinate 0

7. What is the equation of the tangent to the circle defined by $x^2 + y^2 = 11$ at $(3, \sqrt{2})$?

8. Find the equation of the tangent to the circle defined by

$$(x - \sqrt{3})^2 + (y + 3\sqrt{3})^2 = 15 \text{ at } (2\sqrt{3}, -\sqrt{3})$$



C **9.** To find the equation of the tangents with slope $\frac{1}{2}$ to the circle defined by $x^2 + y^2 = 20$:
(a) If the slope of the tangents AB and CD is $\frac{1}{2}$, what is the slope of PQ?
(b) Find the equation of PQ in the form

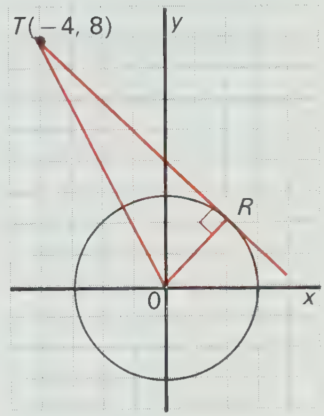
$$y = mx$$

(c) Find the x coordinates of P and Q by substituting this value for y into $x^2 + y^2 = 20$.
(d) Find the corresponding y values by substituting these values for x into the equation for PQ.
(e) Use the slope point form to find the equations of AB and CD.

10. Use the method of question 9 to find the equations of the tangents with slope $\frac{3}{5}$ to the circle defined by $x^2 + y^2 = 34$.

11.5 LENGTH OF A TANGENT TO A CIRCLE

EXAMPLE 1. *Given the circle defined by $x^2 + y^2 = 16$ and the point $T(-4, 8)$ outside the circle, find the length of the tangent TR.*



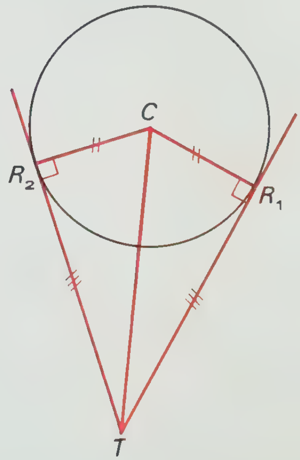
Solution

By definition $\triangle TRO$ is right angled at R

$$\begin{aligned}
\therefore TR^2 &= TO^2 - OR^2 \\
TO^2 &= (-4 - 0)^2 + (8 - 0)^2 \\
&= 16 + 64 \\
&= 80 \\
OR^2 &= 16 \quad (\text{from } x^2 + y^2 = 16) \\
\therefore TR^2 &= 80 - 16 \\
&= 64 \\
\therefore TR &= \sqrt{64} \\
&= 8
\end{aligned}$$

The length of the tangent is 8 units.

EXAMPLE 2. *Find the length of the tangents drawn to the circle with equation $(x - 6)^2 + (y - 2)^2 = 24$ from the point $T(15, 10)$.*



Solution $TR^2 = CT^2 - CR^2$

Centre of circle is $C(6, 2)$ (from the equation)

$$\begin{aligned}
\therefore CT^2 &= (15 - 6)^2 + (10 - 2)^2 \\
&= 81 + 64 \\
&= 145 \\
CR^2 &= 24 \quad (\text{from the equation}) \\
\therefore TR^2 &= 145 - 24 \\
&= 121 \\
\therefore TR &= 11
\end{aligned}$$

The length of the tangent is 11 units.

EXERCISE 11-5

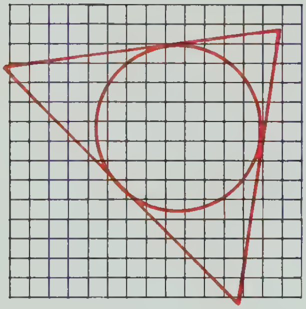
A 1. Give the centre and radius of the circles defined by each of the following equations.

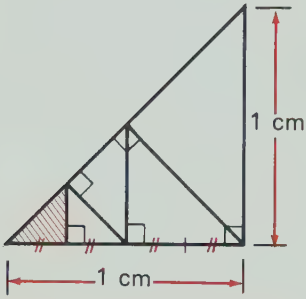
(a) $x^2 + y^2 = 64$ (b) $x^2 + y^2 = 20$
(c) $(x - 3)^2 + (y - 5)^2 = 25$ (d) $(x - 2)^2 + (y + 6)^2 = 36$
(e) $x^2 + (y + 2)^2 = 32$ (f) $(x + 7)^2 + (y - 8)^2 = 100$

B 2. Find the lengths of the following line segments.

(a) $A(3, 5)$ $B(7, 5)$ (b) $C(-2, -7)$ $D(-2, 4)$
(c) $E(7, 8)$ $F(4, 12)$ (d) $G(-3, 6)$ $H(2, -6)$
(e) $J(1, 5)$ $K(6, 15)$ (f) $L(0, 8)$ $M(-6, 0)$

3. Find the length of the tangents drawn from (5, 7) to the circle with the equation $x^2 + y^2 = 64$.





Find the area of the shaded triangle.

4. Find the length of a tangent drawn from $(-4, -3)$ to the circle with the equation $x^2 + y^2 = 13$.

5. Find the length of the tangents drawn from $(8, -3)$ to the circle with the equation $(x - 4)^2 + (y + 7)^2 = 16$.

6. Find the length of the tangent from $(9, 3)$ to the circle defined by $(x + 4)^2 + (y - 3)^2 = 25$.

7. The circle $x^2 + (y - 2)^2 = 18$ is inscribed in the triangle $A(-9, 5)$ $B(3, -7)$, $C(5, 7)$.

(a) Find the lengths of tangents AP and BP and check their sum by finding AB.

(b) Repeat for the other two sides. To save calculations, make use of a fact we have shown about two tangents to a circle from an external point.

8. In Figure 11-2, the point of contact, Q, bisects OR. The circle with equation $(x - 4)^2 + (y - 3)^2 = 9$ is tangent to OR and OX. Find the length of OR.

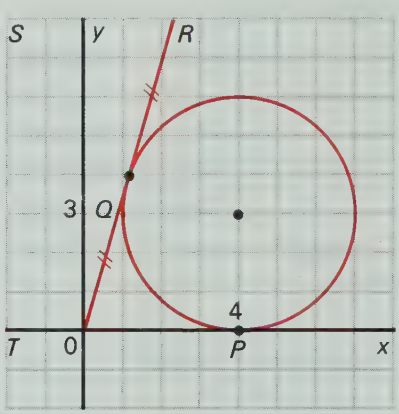


Figure 11-2

C 9. In Figure 11-3, PQR is a tangent to circles with equation $x^2 + y^2 = 1$ and $(x - 5)^2 + y^2 = 9$, where P is the point $(-4, 7)$. By finding the lengths of the tangents PQ and PR find the length of QR.

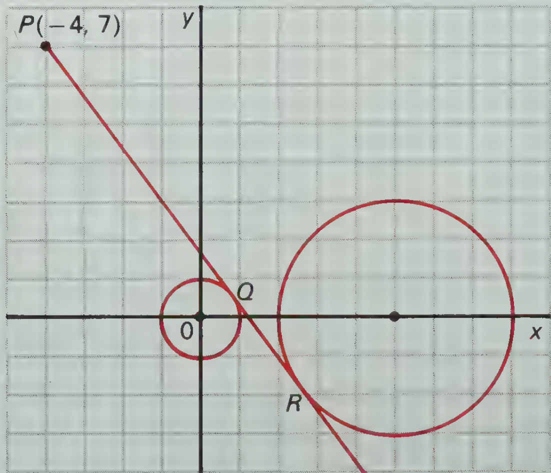


Figure 11-3

10. In Figure 11-4, two tangents PAC and PBD are drawn to the circles $x^2 + y^2 = 5$ and $(x - 3)^2 + (y + 1)^2 = 20$ from the point $(-3, 1)$. Find the lengths of AC and BD. In what ratio does A divide PC?

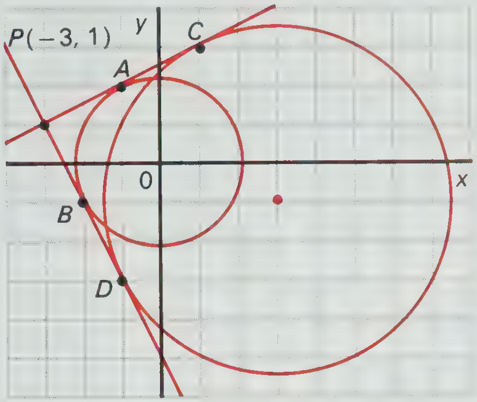


Figure 11-4

11.6 THE RIGHT BISECTOR OF A CHORD

EXAMPLE 1. *Given the circle with equation $(x + 2)^2 + (y - 3)^2 = 13$, show that the right bisector of the chord joining $A(-4, 0)$ and $B(0, 6)$ passes through the centre of the circle.*

Solution 1. To find equation of right bisector of AB:

$$\text{Midpoint of } AB = \frac{-4+0}{2}, \frac{0+6}{2} \qquad \text{Slope } AB = \frac{\triangle y}{\triangle x}$$

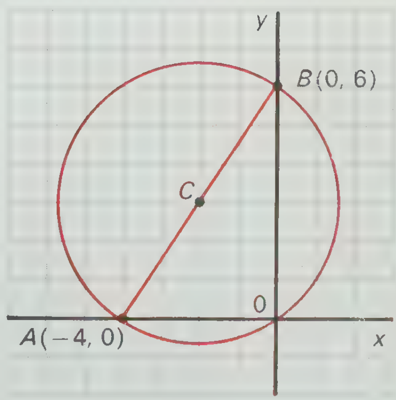
$$= (-2, 3) \qquad = \tfrac{6}{4}$$

$$= \tfrac{3}{2}$$

$$\therefore \text{ Slope of right bisector} = -\tfrac{2}{3}$$

The equation of the right bisector is

$$y - y_1 = m(x - x_1)$$
$$y - 3 = -\tfrac{2}{3}(x + 2)$$
$$3y - 9 = -2x - 4$$
$$2x + 3y - 5 = 0$$



To show that the right bisector passes through the centre of the circle:
Centre of circle $(-2, 3)$ (from equation)
Substitute into $2x + 3y - 5 = 0$

$$\text{LS} = -4 + 9 - 5$$
$$= 0$$
$$= RS$$

$\therefore$ The bisector passes through the centre.

EXAMPLE 2. *Use the property of the right bisector discussed in example 1 to find the equation of the circle passing through P(4, −1), Q(2, 3), R(−2, 5).*

Solution

(i) Consider Chord PQ

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$= (3, 1)$$
$$\text{Slope } PQ = \frac{\triangle y}{\triangle x}$$
$$= \frac{3 + 1}{2 - 4}$$
$$= -\tfrac{4}{2}$$
$$= -2$$

Equation of right bisector

$$y - 1 = \tfrac{1}{2}(x - 3)$$
$$2y - 2 = x - 3$$
$$x - 2y - 1 = 0$$

(ii) Consider Chord QR

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$= (0, 4)$$
$$\text{Slope } QR = \frac{\triangle y}{\triangle x}$$
$$= \frac{5 - 3}{-2 - 2}$$
$$= -\tfrac{2}{4}$$
$$= -\tfrac{1}{2}$$

Equation of right bisector

$$y - 4 = 2(x - 0)$$
$$y - 4 = 2x$$
$$2x - y + 4 = 0$$

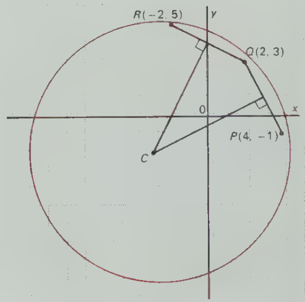
(iii) To find the centre, solve $x - 2y - 1 = 0$ and $2x - y + 4 = 0$

$$x - 2y = 1 \quad (\times 2) \quad 2x - 4y = 2$$
$$2x - y = -4$$
$$\text{(subtract)} \quad -3y = 6$$
$$y = -2$$

Substitute ($y = -2$)

$$x - 2(-2) = 1$$
$$x + 4 = 1$$
$$x = -3$$

$\therefore$ The centre is at $(-3, -2)$



(iv) Radius $r = PC$

$$= \sqrt{(-3 - 4)^2 + (-2 + 1)^2}$$
$$= \sqrt{49 + 1}$$
$$= \sqrt{50}$$
$$\therefore r^2 = 50$$

(v) The equation of the circle is $(x + 3)^2 + (y + 2)^2 = 50$

EXERCISE 11-6

1. Given the circle with equation $x^2 + y^2 = 25$ and points $A(-3, 4)$ and $B(5, 0)$, draw an accurate diagram.
(a) Show that $(-3, 4)$ and $(5, 0)$ satisfy the equation of the circle and hence that A and B lie on the circle.
(b) Find the equation of the right bisector of AB.
(c) Show that it passes through the centre of the circle.

2. Given the circle with equation $(x - 2)^2 + (y + 1)^2 = 10$ and points $A(1, 2)$ and $B(3, 2)$ on the circle, draw an accurate diagram.
(a) Find the slope of the line segment from the centre to the midpoint of AB.
(b) Find the slope of AB.
(c) What relationship exists between the line segments in (a) and (b)?

3. Given the circle with equation $(x - 5)^2 + (y + 3)^2 = 16$ and points $A(9, -3)$ and $B(5, 1)$ on the circle, draw an accurate diagram.
(a) Find the midpoint of AB.
(b) Find the equation of the line through the centre of the circle perpendicular to AB.
(c) Do the coordinates of the point in (a) satisfy the equation of the line in (b)? State a conclusion.

4. Show that $(2, 5)$, $(3, 4)$, and $(-2, 5)$ are on a circle with centre at $(0, 2)$. Find the equation of the circle.

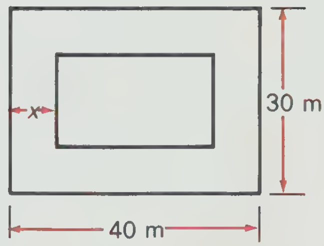
5. Given the points $O(0, 0)$, $P(-4, 0)$, and $Q(0, 6)$,
(a) Find the equations of the right bisectors of OP and OQ.
(b) Find the point of intersection, C, of the right bisectors found in (a).
(c) Find the distance $OC = r$.
(d) Find the equation of the circle passing through OPQ.

6. Find the equation of the circle through $P(8, 0)$, $Q(4, 2)$, and $R(7, 1)$.

7. (a) What are the x-intercepts of $x^2 + y^2 = 169$?
(b) What are the coordinates of A and B, the ends of the diameter along the x-axis of $x^2 + y^2 = 169$?
(c) Does $C(12, 5)$ lie inside, on, or outside the circle $x^2 + y^2 = 169$?
(d) Find the slopes of AC and BC.
(e) How are AC and BC related?
(f) What is the measure of an angle in a semi-circle?

8. Three points $A(-4, 3)$, $B(4, -3)$, and $C(3, 4)$ lie on a circle with centre $(0, 0)$, radius 5.
(a) What special chord is AB?
(b) Find the slopes of AC and BC.
(c) Find the measure of angle ACB.

9. Two circles with equations $(x - 1)^2 + (y - 2)^2 = 9$ and $(x - 7)^2 + (y - 5)^2 = 25$ intersect at $M(2, 5)$ and $N(4, 1)$.
(a) What are the coordinates of the centre, C, of the circle $(x - 1)^2 + (y - 2)^2 = 9$?
(b) What are the coordinates of the centre, K, of the circle $(x - 7)^2 + (y - 5)^2 = 25$?
(c) Find the slope of CK.



A lawn is 30 m by 40 m. How wide a strip is mowed around the outside when the lawn is half done?

(d) Find the slope of *MN*.

(e) How are *MN* and *CK* related?

(f) Find the point of intersection of *MN* and *CK*.

(g) How is the line joining the two centres of two intersecting circles related to the line joining the two points of intersection?

REVIEW EXERCISE

B

1. Find the equation of a circle with centre (0, 0) and:

(a) radius 6 (b) radius 9

(c) passing through $(-4, -3)$ (d) y-intercept 5

(e) x-intercept 7 (f) passing through (1, 1)

2. Find the equation of a circle with centre:

(a) (1, 2) radius 3 (b) (2, 3) radius 5

(c) (0, 4) passing through $(1, -2)$ (d) (0, 1) y-intercept 5

(e) $(7, -2)$ passing through the origin (f) (4, 0) y-intercept 3

3. Sketch the graphs of the following inequalities.

(a) $x^2 + y^2 \geqq 1$ (b) $x^2 + y^2 < 16$ (c) $x^2 + y^2 \leqq 25$

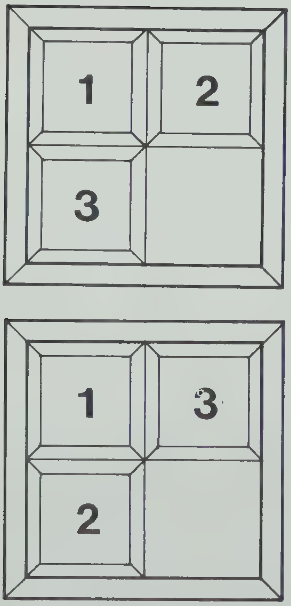
(d) $x^2 + y^2 > 144$ (e) $x^2 + y^2 \leqq 9$ (f) $x^2 + y^2 \geqq 81$

4. (a) Find the equation of a circle having its centre at the origin and passing through the point (1, 2).

(b) Find the equation of the tangent at (1, 2).

5. (a) Find the equation of the circle whose diameter has endpoints $A(3, -2)$ and $B(-3, 2)$.

(b) Find the equations of the tangents at *A* and *B*.



Can you slide the numbers in the second box into the order of the numbers in the first box without removing or overlapping any?

6. Find the equations of the following tangents:
(a) To $x^2 + y^2 = 29$ at $(-2, 5)$
(b) To $x^2 + y^2 = 36$ at $(0, 6)$
(c) To $(x - 4)^2 + (y + 3)^2 = 13$ at $(1, -5)$
(d) To $(x + 3)^2 + y^2 = 40$ at $(3, -2)$

7. Find the lengths of the following tangents.
(a) From $(14, 12)$ to $x^2 + y^2 = 100$
(b) From $(-8, 5)$ to $x^2 + y^2 = 40$
(c) From $(4, 7)$ to $x^2 + (y - 3)^2 = 16$
(d) From $(6, 0)$ to $(x + 3)^2 + y^2 = 56$

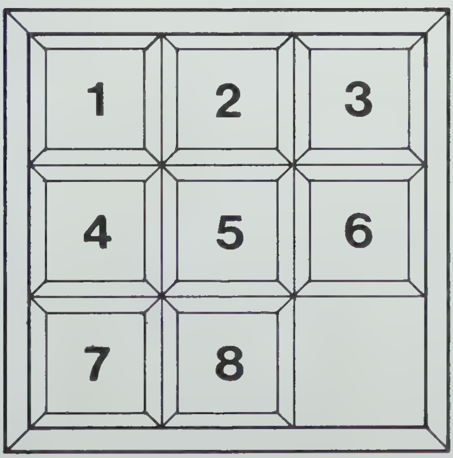
8. (a) Draw the graphs of $x^2 + y^2 = 25$ and $(x - 4)^2 + y^2 = 1$.
(b) State the centre, C, of $x^2 + y^2 = 25$ and the centre, K, of $(x - 4)^2 + y^2 = 1$.
(c) State the coordinates of I, the point of intersection of the two circles.
(d) Find slope CI and slope KI.
(e) What kind of points are C, K and I?

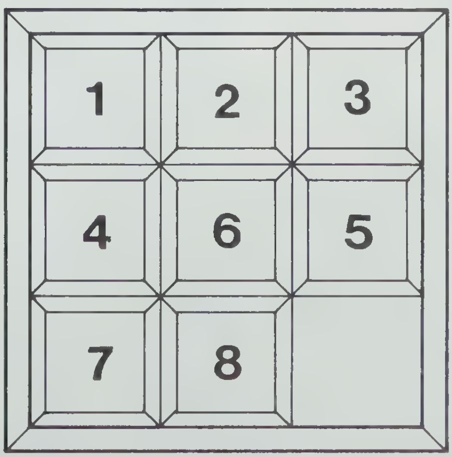
9. (a) Find the length of the chord cut from the line $x - y = 1$ by the circle $x^2 + y^2 = 25$.
(b) Find the midpoint, M, of the chord.
(c) Find the slope MO, where O is the centre of the circle.
(d) Find the slope of the chord.

10. How long is a chord of the circle $x^2 + y^2 = 100$ if it is 6 units from the centre?

11. Find the equation of the circle passing through $(1, 2)$, $(-3, 2)$, and $(1, 8)$.

Can you slide the numbers in the second box into the order of the numbers in the first box without removing or overlapping any?





Day of Month	January Day of Year	January Decimal Equivalent	February Day of Year	February Decimal Equivalent
1	1	0.0027	32	0.0877
2	2	0.0055	33	0.0904
3	3	0.0082	34	0.0932
4	4	0.0110	35	0.0959
5	5	0.0137	36	0.0986
6	6	0.0164	37	0.1014
7	7	0.0192	38	0.1041
8	8	0.0219	39	0.1068
9	9	0.0247	40	0.1096
10	10	0.0274	41	0.1123
11	11	0.0301	42	0.1151
12	12	0.0329	43	0.1178
13	13	0.0356	44	0.1205
14	14	0.0384	45	0.1233
15	15	0.0411	46	0.1260
16	16	0.0438	47	0.1288
17	17	0.0466	48	0.1315
18	18	0.0493	49	0.1342
19	19	0.0521	50	0.1370
20	20	0.0548	51	0.1397
21	21	0.0575	52	0.1425
22	22	0.0603	53	0.1452
23	23	0.0630	54	0.1479
24	24	0.0658	55	0.1507
25	25	0.0685	56	0.1534
26	26	0.0712	57	0.1562
27	27	0.0740	58	0.1589
28	28	0.0767	59	0.1616
29	29	0.0795		
30	30	0.0822		
31	31	0.0849		

Figure 12-1

REVIEW AND PREVIEW TO CHAPTER 12

BORROWING MONEY

In chapter 12 we shall be discussing the borrowing of money. The following definitions should be recalled from previous work.

Principal (P): the sum of money borrowed.

Interest (I): the money charged for the use of the principal.

Rate per annum (r): the interest charged per year expressed as a percentage of the principal.

Time (t): the length of time on which interest is charged.

Amount (A): the sum of money to be repaid, including interest and principal.

PERCENTAGES

Most work with finance involves percentages in some form or another. Calculate the following using your slide rule or calculator.

EXERCISE 1

1. 2% of \$15. **2.** 3% of \$169. **3.** 3% of \$534.
4. 4% of \$9640. **5.** 1% of \$274. **6.** 6% of \$753.
7. 5% of \$43. **8.** 7% of \$971. **9.** 8% of \$846.
10. 8% of \$536. **11.** 9% of \$634. **12.** $4\frac{1}{2}$% of \$71.
13. 12% of \$864. **14.** $10\frac{1}{4}$% of \$579. **15.** $3\frac{1}{2}$% of \$654.
16. 5% of \$805. **17.** 15% of \$1650. **18.** $12\frac{1}{2}$% of \$395.

FRACTIONS OF A YEAR

Since interest is usually given in per annum rates, time must be expressed as a fraction of a year. The table shown in Figure 12-1 gives any number of days as a decimal fraction of a year. The full table appears in the appendix.

Where the time is given in months, assume 30 d/mo.*

EXERCISE 2

Express the following as fractions of a year.

1. 33 d **2.** 63 d **3.** 93 d
4. 5 mo. **5.** 2 mo. **6.** 7 mo.
7. 12 w. **8.** 26 w. **9.** 60 d
10. 90 d **11.** 120 d **12.** 3 w., 2 d
13. 5 w., 2 d **14.** 8 w. **15.** 10 mo.
16. 30 mo. **17.** 4 w., 2 d **18.** 6 w., 3 d
19. 48 h **20.** 72 h

* *See page opposite Contents page.*

DAYS AND DATES

The same table (Figure 12-1) can be used to find the number of days, and hence the fraction of a year, between two dates.

EXAMPLE 1. *Find the number of days between 04-24 and 06-16. Express your answer as a fraction of a year.*

Solution

06-16 : day 167 (from table)
04-24 : day 114 (from table)
Number of days = 167 − 114
= 53
53 d = 0.1452 a (from table)

EXAMPLE 2. *A loan is made on 04-23 and is to be repaid in 93* d. *Find the due date.*

Solution

04-23 : day 113
113 + 93 = 206
day 206 = 07-25

The due date is 07-25.

EXERCISE 3

Find the number of days between the following dates and the fraction of a year.

1. 01-06,03-16
2. 02-14,05-30
3. 02-28,07-19
4. 03-07,09-21
5. 04-22,11-01
6. 05-12,09-08
7. 07-15,12-20
8. 10-03,11-17
9. 11-17,03-18
10. 10-29,03-13
11. 12-20,05-06
12. 11-01,01-08

Find the following dates.

13. 30 d after 02-06
14. 63 d after 03-04
15. 25 d after 04-28
16. 93 d after 05-17
17. 120 d after 06-08
18. 57 d after 08-11
19. 75 d after 11-07
20. 150 d after 12-15

ESTIMATING BY THE 6% FOR 60 DAYS METHOD

Sixty days is about a sixth of a year. If *P* dollars are invested at 6% for 60 d,

March		April	
Day of Year	Decimal Equivalent	Day of Year	Decimal Equivalent
60	0.1644	*91*	0.2493
61	0.1671	*92*	0.2521
62	0.1699	*93*	0.2548
63	0.1726	*94*	0.2575
64	0.1753	*95*	0.2603
65	0.1781	*96*	0.2630
66	0.1808	*97*	0.2658
67	0.1836	*98*	0.2685
68	0.1863	*99*	0.2712
69	0.1890	*100*	0.2740
70	0.1918	*101*	0.2767
71	0.1945	*102*	0.2795
72	0.1973	*103*	0.2822
73	0.2000	*104*	0.2849
74	0.2027	*105*	0.2877
75	0.2055	*106*	0.2904
76	0.2082	*107*	0.2932
77	0.2110	*108*	0.2959
78	0.2137	*109*	0.2986
79	0.2164	*110*	0.3014
80	0.2192	*111*	0.3041
81	0.2219	*112*	0.3068
82	0.2247	*113*	0.3096
83	0.2274	*114*	0.3123
84	0.2301	*115*	0.3151
85	0.2329	*116*	0.3178
86	0.2356	*117*	0.3205
87	0.2384	*118*	0.3233
88	0.2411	*119*	0.3260
89	0.2438	*120*	0.3288
90	0.2466		

Figure 12-1

May		June	
Day of Year	Decimal Equivalent	Day of Year	Decimal Equivalent
121	0.3315	*152*	0.4164
122	0.3342	*153*	0.4192
123	0.3370	*154*	0.4219
124	0.3397	*155*	0.4247
125	0.3425	*156*	0.4274
126	0.3452	*157*	0.4301
127	0.3479	*158*	0.4329
128	0.3507	*159*	0.4356
129	0.3534	*160*	0.4384
130	0.3562	*161*	0.4411
131	0.3589	*162*	0.4438
132	0.3616	*163*	0.4466
133	0.3644	*164*	0.4493
134	0.3671	*165*	0.4521
135	0.3699	*166*	0.4548
136	0.3726	*167*	0.4575
137	0.3753	*168*	0.4603
138	0.3781	*169*	0.4630
139	0.3808	*170*	0.4658
140	0.3836	*171*	0.4685
141	0.3863	*172*	0.4712
142	0.3890	*173*	0.4740
143	0.3918	*174*	0.4767
144	0.3945	*175*	0.4795
145	0.3973	*176*	0.4822
146	0.4000	*177*	0.4849
147	0.4027	*178*	0.4877
148	0.4055	*179*	0.4904
149	0.4082	*180*	0.4932
150	0.4110	*181*	0.4959
151	0.4137		

Figure 12-1

$$I = Prt$$

$$I = P \times 0.06 \times \tfrac{1}{6}$$

$$= 0.01P \text{ or } 1\% \text{ of the principal.}$$

If I borrow \$350 for 60 d at 6%, the interest is approximately 1% of \$350 or \$3.50.

Once this relation is learned it is simple to estimate for 3%, 12% or 18% by halving, doubling, or tripling. Similarly, for times of 30 d, 90 d (3 mo.), or 120 d (4 mo.) mental adjustments can be made.

EXAMPLE 1. *Approximate the interest on \$585 at 12% for 60* d.

Solution 6% for 60 d gives 1%. Therefore 12% for 60 d gives 2%.

Interest = 0.02 × \$585.00
= \$11.70

EXERCISE 4

Estimate the interest for the following loans. (Answers are given for even-numbered questions only.)

1. \$725 at 6% for 60 d
2. \$3570 at 6% for 60 d
3. \$139 at 3% for 60 d
4. \$87 at 3% for 60 d
5. \$5900 at 12% for 60 d
6. \$45 at 12% for 60 d
7. \$426 at 6% for 30 d
8. \$25 at 6% for 90 d
9. \$5062 at 12% for 30 d
10. \$75 at 8% for 60 d
11. \$523 at 8% for 120 d
12. \$236 at 8% for 30 d
13. \$7859 at 10% for 60 d
14. \$258 at 10% for 120 d

Solve for the variable (Use $\pi = 3.141\ 59$).

1. $A = \pi(46.37)^2$
2. $A = \pi(2.473)^2$
3. $A = \pi(146.2)^2$
4. $A = \pi(0.7134)^2$
5. $A = \pi(0.0758)^2$
6. $A = \pi(84.61)^2$
7. $A = \pi(957.3)^2$
8. $A = \pi(4632)^2$
9. $A = \pi(66.66)^2$
10. $A = \pi(0.4813)^2$

Borrowing Money

CHAPTER 12

12.1 FINANCING WITH SIMPLE INTEREST

The Basic Ideas of Simple Interest

If a homeowner has an unoccupied house which he does not wish to sell, he will probably rent it. The house is depreciating, and taxes must still be paid, so the owner must get revenue to pay these expenses. If the landlord feels there is likelihood of the tenants damaging the property, he will add to the rent an amount sufficient to cover his estimated repairs.

If a man has money that he is not using, he will be wise to rent it out. In this case we say he is lending or investing it. It is true that there are no recurring taxes on money and old bills have the same value as new bills, but the money is losing value, since goods are becoming more expensive. While the investor does not worry that the money may be damaged, he is concerned that it will be repaid. If the borrower has a house, car, or securities which he can put up as surety for the loan, the interest rate will be lower than if there is little or no guarantee that the loan will be repaid.

Two equations used in simple interest are:

Amount = Principal + Interest. $A = P + I$
Interest = Principal × Rate × Time. $I = Prt$

By substitution,

$$A = P + Prt$$
$$= P(1 + rt)$$
$$\therefore P = \frac{A}{1 + rt}$$

EXAMPLE 1. *Find the interest and amount to be repaid if \$570.00 is borrowed for 63* d *at 12%.*

Solution $P = \$570 \qquad r = 12\% = 0.12 \qquad t = 63\text{ d} = 0.1726\text{ a}$

$$I = Prt$$
$$I = \$570 \times 0.12 \times 0.1726$$
$$= \$11.805$$
$$\doteq \$11.81$$
$$A = P + I$$

Day of Month	July: Day of Year	July: Decimal Equivalent	August: Day of Year	August: Decimal Equivalent
1	*182*	0.4986	*213*	0.5836
2	*183*	0.5014	*214*	0.5863
3	*184*	0.5041	*215*	0.5890
4	*185*	0.5068	*216*	0.5918
5	*186*	0.5096	*217*	0.5945
6	*187*	0.5123	*218*	0.5973
7	*188*	0.5151	*219*	0.6000
8	*189*	0.5178	*220*	0.6027
9	*190*	0.5205	*221*	0.6055
10	*191*	0.5233	*222*	0.6082
11	*192*	0.5260	*223*	0.6110
12	*193*	0.5288	*224*	0.6137
13	*194*	0.5315	*225*	0.6164
14	*195*	0.5342	*226*	0.6192
15	*196*	0.5370	*277*	0.6219
16	*197*	0.5397	*228*	0.6247
17	*198*	0.5425	*229*	0.6274
18	*199*	0.5452	*230*	0.6301
19	*200*	0.5479	*231*	0.6329
20	*201*	0.5507	*232*	0.6356
21	*202*	0.5534	*233*	0.6384
22	*203*	0.5562	*234*	0.6411
23	*204*	0.5589	*235*	0.6438
24	*205*	0.5616	*236*	0.6466
25	*206*	0.5644	*237*	0.6493
26	*207*	0.5671	*238*	0.6521
27	*208*	0.5699	*239*	0.6548
28	*209*	0.5726	*240*	0.6575
29	*210*	0.5753	*241*	0.6603
30	*211*	0.5781	*242*	0.6630
31	*212*	0.5808	*243*	0.6658

Figure 12-1

$$I = \$570.00 + \$11.81$$
$$= \$581.81$$

The interest charged is \$11.81 and the amount repaid is \$581.81.

EXERCISE 12-1

A

1. (a) Define:
   (i) amount (ii) interest (iii) principal
   (iv) rate (v) time

   (b) Give three formulas which describe relations between the quantities defined in (a).

B

2. Solve the formula $I = Prt$ for r and t.

3. Calculate the interest and the amount repaid in each case.
   (a) \$4000 at 7% for 6 mo. (b) \$520 at 8% for 3 mo.
   (c) \$5280.00 at 9% for 256 d (d) \$750 at 9% for 8 mo.
   (e) \$429.65 at $5\frac{1}{2}$% for 5 w. (f) \$2500 at 10% for 7 mo.

4. Find the missing quantity in each line.

	I	P	r	t
(a)		\$200	6%	6 mo.
(b)		\$1200	$5\frac{1}{2}$%	1 a
(c)	\$65		5%	3 mo.
(d)	\$90		12%	1 a
(e)	\$640	\$16 000		6 mo.
(f)	\$31.50	\$350		9 mo.
(g)	\$2.25	\$90	6%	
(h)	\$3750	\$50 000	$7\frac{1}{2}$%	

5. Calculate the amounts to be repaid to the nearest cent. Use a calculator if available.
   (a) \$375.00 at 6% for 40 d (b) \$539.00 at 8% for 60 d
   (c) \$5280.00 at 9% for 256 d (d) \$125.25 at $6\frac{1}{2}$% for 93 d
   (e) \$429.65 at $5\frac{1}{2}$% for 5 w. (f) \$7348.00 at $8\frac{1}{4}$% for 150 d

6. The Cedar-wood Canoe Company borrowed working capital at 7% to keep the plant producing over the slack winter months. After 170 d they repaid the total loan with a payment of \$10 326.03. How much had they borrowed?

7. What principal must be loaned out at $5\frac{1}{2}$% to give a return of \$5000 after 195 d?

8. The Extension Rule Company borrowed \$15 000 for 130 d and repaid \$15 293.84. What rate of simple interest was charged?

9. J. P. Howard endowed his alma mater with a scholarship which amounted to the annual income from \$25 000 invested at $6\frac{3}{4}$%. What was the annual cash value of the scholarship?

10. South Western Trust pays $5\frac{1}{2}$%/a on deposits in its savings accounts. At the end of 6 mo. the interest is calculated and added to the account. The interest for the next period is calculated on the new balance. If you deposit \$500 in an account and leave it for 2 a, what balance will be in the account?

11. A loan of $500 was taken out on 06-15 and repaid on 08-30 the same year. If interest was charged at 12%, how much was repaid?

12. $1500 was borrowed for 93 d on 04-23. If interest was charged at $9\frac{1}{2}$%, what amount was repaid? When was it due?

13. $750.00 was borrowed on 03-03 and $755.42 was repaid on 04-05.
(a) How much interest was charged?
(b) What rate of interest was charged?

14. $5000 was borrowed on 02-04 at $7\frac{1}{2}$%. The loan was repaid by a payment of $5154.13. When was it made?

Would you pay $10 for a chocolate bar, $32 000 for a standard automobile, or $420 for a bicycle? In each case a misplaced decimal has rendered the figure ridiculous. It is important that you learn to recognize gross errors when dealing with finances.

12.2 THE COST OF BORROWING MONEY EFFECTIVE ANNUAL RATES

EXAMPLE 1. *Mr. Jones borrowed $1500 to pay the difference between his trade-in allowance and the cost of a new car. The loan was repaid by 24 payments of $72 beginning the month after the loan was made. What rate of interest was charged?*

Solution We can reason the approximate rate of interest by considering that at the time the loan was made the principal was $1500 and at the time of the last payment the principal was $72 therefore:

AVERAGE PRINCIPAL METHOD

$$\text{Average Principal, } P_{av} = \frac{\$1500 + \$72}{2} = \$786$$

$$\text{Interest} \quad I = A - P = 24 \times \$72 - \$1500 = \$1728 - \$1500 = \$228$$

$$\text{Rate} \quad r = \frac{I}{P_{av} \cdot t} = \frac{228}{786 \times 2} = 0.145$$

The interest rate is 14.5%.

FORMULA METHOD
Where greater accuracy is required we can use the formula:

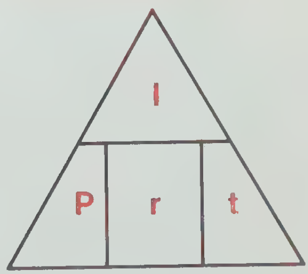
$$r = \frac{2NI}{P(n+1)}$$

where r is the effective annual rate
N is the number of payment periods in 1 a
I is the interest charged
P is the principal
n is the total number of payments

In this case $r = \frac{2(12)(228)}{1500 \times 25} = 14.6$

The rate is 14.6%

Cover the one you want to find its expression.



When buying financing services, you should shop as carefully as you do for the articles you buy. The way to check the cost of finance is to compare the effective annual rates.

EXAMPLE 2. *Mr. Staniuk borrowed \$2500 from his bank to put an addition on his home. The loan was to be repaid by 30 monthly instalments of \$95. Calculate the effective annual rate being charged.*

Solution $n = 30,\ A = 30 \times \$95 = \$2850,\ P = \$2500,\ I = \$350,\ N = 12$

(i) Average principal method

$$\text{Average principal} = \frac{\$2500 + \$95}{2}$$
$$= \$1297.50$$
$$\therefore r = \frac{350}{1297.50 \times \frac{30}{12}}$$
$$= 0.108 = 10.8\%$$

(ii) Formula method

$$r = \frac{2NI}{P(n+1)}$$
$$= \frac{2(12)(350)}{2500(31)}$$
$$= 0.108$$
$$= 10.8\%$$

EXERCISE 12-2

B

1. Find the interest charged and the effective annual rate of interest to the nearest $\frac{1}{2}$% for each of the following.

	Principal	Monthly Payments
(a)	\$100	12 of \$8.79
(b)	\$500	18 of \$30.00
(c)	\$1 000	24 of \$47.00
(d)	\$3 000	36 of \$96.84
(e)	\$12 500	26 of \$500.00
(f)	\$20 000	120 of \$250.00

2. Easy Loan Company advertised a loan of \$500 repaid by 12 easy payments of \$45.

Cash-4-U advertised a loan of \$600 repaid by 12 payments of \$54.

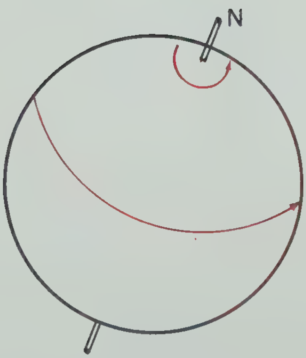
Calculate the interest rate charged by each company.

3. A "Fly now, pay later" plan offered a \$375 air fare at \$22.50/mo. for 18 mo. What interest rate was being charged?

4. Mr Plaunt borrowed \$600 to pay medical bills. Interest was charged at 6%/a on the total amount borrowed for 9 mo. The loan was repaid by nine equal monthly payments beginning 1 mo. after the loan was made. Calculate:

(a) the amount of each payment. (b) the effective annual rate.

5. Mr. Fortin purchased a refrigerator, stereo, chesterfield, and automobile on time payments. Mr. Fortin's payments fell in arrears when he was laid off temporarily. Now that he is back at work he feels the only way he can catch up is to consolidate his debts and pay them off by means of a single loan which will reduce monthly payments, but extend the time of the debt. To do this he borrows \$3000 to be repaid over 30 mo. with payments of \$113.80. What interest rate is he being charged?



The radius of the earth is about 6400 km, and it makes 1 r/24 h. How fast is a person standing at the equator traveling relative to the centre of the earth?

12.3 INSTALMENT BUYING

EXAMPLE 1. *A sign in a jewelry store advertises watches. On enquiring, it is found that there will be 35 weekly payments. Find:*
(a) the instalment price;
(b) the instalment charges;
(c) the effective annual rate.

WATCHES
$55
Only $10⁰⁰ down
$1.50 PER WEEK

Solution

(a) $$\text{Instalment price} = \$10 + (35 \times \$1.50) = \$10 + \$52.50 = \$62.50$$

(b) $$\text{Instalment charge} = \$62.50 - \$55 = \$7.50$$

(c) $$\text{Amount financed} = \$55.00 - \$10.00 = \$45$$

$$\text{Effective annual rate} = \frac{2NI}{P(n+1)} = \frac{2(52)(7.50)}{45(36)}$$

$$r = 48\%$$

R is the effective annual rate.
N is the number of payment periods in 1 a.
I is the interest charged.
P is the principal.
n is the total number of payments.

EXERCISE 12-3

B **1.** Find the instalment charges and the rate of interest charged on each of the following articles.

Article	Cash Price	Down Payment	Monthly Payments
(a) Radio	$68.50	$10	10 of $6.50
(b) Refrigerator	$344.95	$0	24 of $18
(c) Boat and Motor	$762.50	$80	24 of $30
(d) Automobile	$3789.50	$1500	30 of $88.50
(e) Tool Set	$125.80	$25	12 of $9.80
(f) Television Set	$565.90	$50	36 of $16.50

2. Easy-built Furniture Company advertises: "Living Room Set. Five Pieces of Furniture to Grace Your Living Room $499.50. Only $50 down and $25.50 a month for 20 mo."
(a) What service charge is being added to the cash price?
(b) What rate is being charged?

3. Mr. Cartwright bought a tricycle for his son's birthday. He paid $9 cash and $5/mo. for 6 mo. The cash price was $37.95. What rate of interest was charged?

4. A bench saw for a home workshop costs $185.60. It can be bought for $45.60 down and 12 monthly payments of $12.25.
(a) What are the finance charges?
(b) What rate of interest is being charged?

5. What service charge is being paid on a fur coat, cash price $379.50, sold for $99.50 down and 30 monthly payments of $10.50?

6. A sewing machine sells for $119 cash or $25 down and 12%/a service charge on the total balance financed. What amount will discharge the

debt in 12 monthly payments?

7. A motorcycle may be bought for $59.50 down and $15/mo. for 18 mo. If the finance charges are $30.55, what is the cash price?

8. A colour television set will cost $562 cash. It can be bought with a $50 allowance for a trade-in in any condition and $22.25/mo. for 24 mo.
(a) What is the cash price with trade?
(b) What are the instalment charges?

9. Modern Furniture Company advertises a bed and dresser set for $50 down and $17.50/mo. for 30 mo. On enquiring, you find there is a 7% discount for cash purchases. What is the cash price?

10. Find the total cost and finance charges on a motorcycle sold for $300 down and $45/mo. for 30 mo. The cash price is $1500. What rate of interest is being charged?

11. A house is advertised at $47 900 with $4500 down. How much is being financed? If the effective annual rate is 9% and there are 240 monthly payments (a 20 a mortgage), use the formula $r = \frac{2NI}{P(n+1)}$ to calculate the total interest charged.

12. A loan of $2500 is repaid by 30 monthly payments. If the effective annual rate is 10.5% use the formula $r = \frac{2NI}{P(n+1)}$ to calculate I, the total interest charged. Add the interest to the principal borrowed and divide by 30 to find the monthly payment.

12.4 PAYING PRINCIPAL PLUS ACCRUED INTEREST

With some forms of time payment plans, the borrower agrees to pay each month a fixed amount from the principal, plus all the interest which has accrued over the past month.

EXAMPLE 1. *Miss Jones required $300 to put a new engine in her automobile. She borrowed this amount from her credit union, agreeing to repay $50 plus 1% interest per month on the unpaid balance. The following repayment schedule was drawn up.*

Name: Miss F. Jones
Date of loan: 1976-11-15
Principal: $300
Rate: 1%/mo. on unpaid balance

Payment No.	Date	Principal	Interest	Payment on Principal	Total Payment
1	12-15	$300	$3	$50	$53
2	01-15	$250	$2.50	$50	$52.50
3	02-15	$200	$2	$50	$52
4	03-15	$150	$1.50	$50	$51.50
5	04-15	$100	$1	$50	$51
6	05-15	$50	$0.50	$50	$50.50

The interest rate of 1%/mo. could be described as being 12%/a. In describing it this way, we would say that the nominal annual rate was 12%. If the effective annual rate were 12%, the interest would be the same, but none of it would be paid until the end of each year of the duration of the loan. Since the interest is being paid at the end of each month in this case, a nominal annual rate of 12% is slightly higher than an effective annual rate of 12%.

EXERCISE 12-4

1. A loan of \$800 from a credit union is to be repaid at the rate of \$100/mo. plus interest of 1%/mo. on the unpaid balance. Draw up a schedule of payments and calculate the total interest charged.

2. A loan of \$600 is repaid at the rate of \$75/mo. plus interest of 1%/mo. on the unpaid balance. Draw up a schedule of payments and calculate the total interest charged.

3. A loan for \$450 from a loan agency is to be repaid at a rate of \$90/mo. with interest charged at $1\frac{1}{2}$%/mo.
(a) Find the total interest charged.
(b) What is the nominal annual rate?

4. Mr. Camelot required \$2550 for a new car. He arranged a loan through his credit union. The principal was to be repaid in 30 equal monthly payments plus $\frac{3}{4}$ of 1% interest charged monthly on the unpaid balance. Draw up a payment schedule for the first three and last two payments. Suggest reasons why an equal payment plan would be more popular.

5. Mrs. Wilson borrowed \$650 from her credit union for a colour television set. She repaid the principal in twelve equal payments plus interest of 1%/mo. on the unpaid balance. At the end of the year the credit union declared a rebate of 15% of the interest paid. What was the total interest paid on the loan? Calculate the nominal rate.

6. A loan of \$800 is to be repaid at \$100/mo. plus interest at $1\frac{1}{2}$%/mo. Find the total interest paid. What is the nominal annual rate?

7. A loan of \$900 is repaid over 6 mo. with interest charged at $1\frac{1}{4}$%/mo. on the unpaid balance. Find the total interest paid in each of the following cases.
(a) Repay \$200/mo. for 3 mo. then
\$100/mo. for 3 mo.
(b) Repay \$150/mo. for 6 mo.
(c) Repay \$100/mo. for 3 mo. then
\$200/mo. for 3 mo.

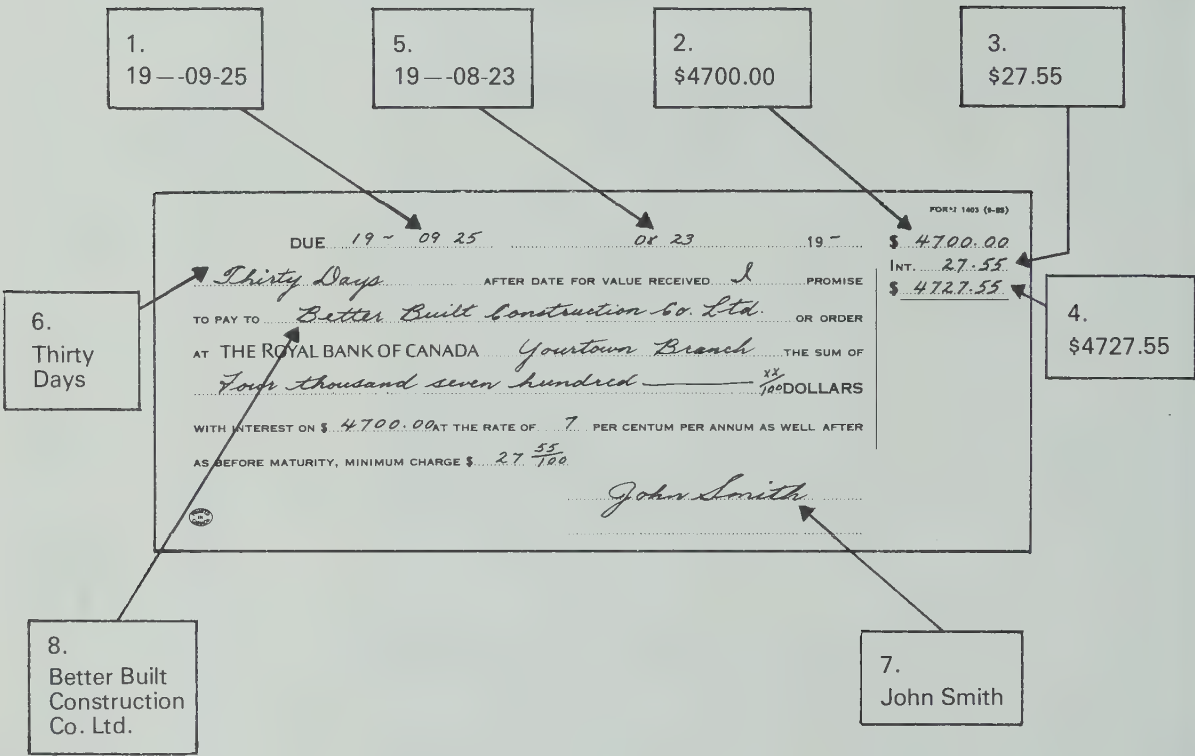
8. \$600 is borrowed for 1 a. Compare the total interest and the annual rate charged in each of the following repayment plans.
(a) \$53.32/mo.
(b) \$50/mo. plus interest at 1%/mo. on the unpaid balance.

The earth travels in an approximately circular orbit about the sun. The radius is about 150 000 000 km and it makes 1 revolution in 365.24 d. How fast is the earth traveling around the sun in km/h?

12.5 BORROWING ON A TIME NOTE

When a debt is to be paid after a specified time the agreement signed by the debtor is called a promissory time note.

The following example will illustrate the main features of a note.



1. The *due date* is 19—-09-25, 30 d plus 3 d grace after the date of the note.
2. The *face* of the note is the principal, $4700.
3. The *interest* is shown separately and is calculated on 30 d plus 3 d grace allowed on all notes by Canadian law.
4. The *maturity value* of the note is the face plus interest. It is the amount that must be paid to redeem the note.
5. The *date* of the note is 19—-08-23, the date the loan was made.
6. The *time* of the note is 30 d.
7. The *maker* of the note is John Smith, who owes the debt.
8. The *payee* is Better Built Construction Company Limited.

EXAMPLE 1. *Miss Schwartz makes a 90-d note for $350 payable to Locke and Keyes Hardware. The note bears interest at 8% before as well as after maturity and is made on 19—-02-06. Find the due date and the maturity value.*

Solution

To find the due date:

Time plus three days grace is 93 d.

02-06 is day	37	(From tables)
	+93	
	130	
05-10 is day 130.		(From tables)

To find the maturity value:

$$93 \text{ d} = 0.2548 \text{ a} \qquad \text{(From table)}$$

$$\begin{aligned} I &= Prt \\ &= \$350 \times 0.08 \times 0.2548 \\ &= \$7.13 \end{aligned}$$

$$\begin{aligned} \text{Maturity value} &= \$350.00 + \$7.13 \\ &= \$357.13 \end{aligned}$$

EXERCISE 12-5

1. In return for her used automobile, Mrs. I. Carman gave Miss U. Sellers a 30-d note for $1200. No interest was charged. If the purchase was made on 03-12:

(a) when did the note fall due?

(b) what was the maturity value of the note?

(c) what risks did Miss Sellers take in accepting the note?

2. Mr. Wilson made a private loan of $7500 to Mr. Marshall on 03-23. Mr. Marshall gave a 30-d note for the amount, bearing interest at $8\frac{1}{2}\%$. Complete the following table in your notes.

Maker		Face	
Payee		Interest rate	
Date of note		Interest	
Due date		Maturity value	

3. Find the due date and the maturity value for the following notes.

	Date	Time	Face	Interest Rate
(a)	01-11	30 d	$275	$7\frac{1}{2}$%
(b)	03-12	60 d	$536	7%
(c)	04-07	80 d	$2760	6%
(d)	05-01	20 d	$730	8%
(e)	06-25	10 d	$5800	7%
(f)	08-19	90 d	$7500	$8\frac{1}{2}$%
(g)	09-04	30 d	$180	$6\frac{1}{2}$%
(h)	10-02	25 d	$300	7%
(i)	11-15	12 d	$5250	6%
(j)	12-20	30 d	$1800	8%

Don't forget 3 d grace.

4. Draw up a note bearing $10\frac{1}{2}$% interest promising to pay R. Quinn the sum of $250.00 thirty days from this date.

5. Draw up a note bearing 12% interest promising to pay J. Winsome $425 ninety days from this date.

12.6 DISCOUNTING NOTES

When a note is discounted, the charge for financing is calculated on the value of the note at maturity. If it is a noninterest-bearing note the value will be the face of the note. A note is usually discounted when the payee needs money before the due date of the note and the note is sold to a third party such as a bank.

Discount = Amount × Rate × Time

$$\boxed{D = Art}$$

Proceeds = Amount − Discount

$$\boxed{Pr = A - D}$$

EXAMPLE 1. *Mr. Campbell required about $500 for 60* d. *He gave the bank a note for $500 which they discounted at $7\frac{1}{2}$%. What were the proceeds of the note?*

Solution The maturity value of the note is $500. Therefore the amount of the loan is $500. The discount is calculated on the $500 in the same manner as interest.

63 d = 0.1726 a

$$\begin{aligned} D &= Art \\ &= 500 \times 0.075 \times 0.1726 \\ &= 6.47 \end{aligned}$$

The discount is $6.47. The proceeds of the note, the amount Mr. Campbell actually receives, will be $500 − $6.47 = $493.53.

EXAMPLE 2. *A farmer paid for seed with a noninterest-bearing 120-*d *note for $250 on 04-26. The seed merchant needed money to pay his own accounts, so on 05-30 he took the note to the bank, which discounted it at 6%. What proceeds did the merchant receive?*

Solution

Date of Note Discount date
Time Scale → 04-26 05-30 ←Discount period→Due date

Value Scale→ Face Proceeds←——Discount——→Maturity value

Number of days from date of note to discount date:

04-26 — 04-30	4 d	or	05-30 = day	150
04-30 — 05-30	30 d		04-26 = day	116
	34 d			34 d

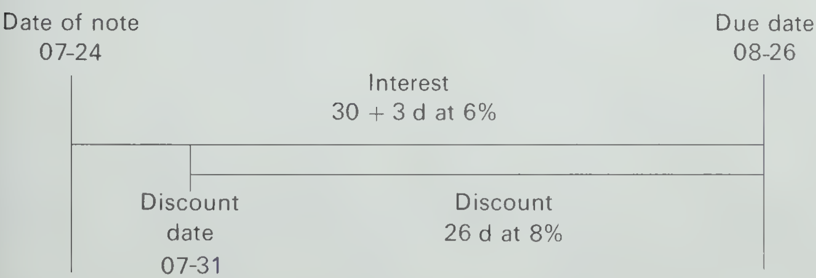
Discount period 123 − 34 = 89

$$\begin{aligned} D &= Art \\ &= \$250 \times 0.06 \times 0.2438 \\ &= \$3.66 \\ Pr &= A - D \\ &= \$250 - \$3.66 \\ &= \$246.34 \end{aligned}$$

The proceeds from the note were $246.34

EXAMPLE 3. *A carpenter agreed to remodel the Croghans' kitchen for $750. On 07-24, the carpenter accepted a 30* d *note bearing interest at 6% for full payment. On 07-31, the carpenter had to meet bills at the lumber company or be charged interest, so the Croghans' note was taken to the bank where it was discounted at 8%. What proceeds did the carpenter receive?*

Solution The discount will be calculated on the value of the note at maturity. Therefore finding the proceeds is a four-step calculation.



Step (1): $I = Prt$ 33 d = 0.0904 a

$= \$750 \times 0.06 \times 0.0904 = \$4.07$

Step (2): Maturity value = face value + interest

$= \$750.00 + \$4.07$
$= \$754.07$

26 d = 0.0712 a

Step (3): $D = Art$
$= \$754.07 \times 0.08 \times 0.0712$
$= \$4.29$

Step (4): Proceeds = maturity value − discount
$= \$754.07 - \$4.29$
$= \$749.78$

EXERCISE 12-6

B

1. Calculate the due date and the proceeds from the following non-interest-bearing notes.

	Date	Face	Discount Rate	Time
(a)	06-04	$650	8%	20 d
(b)	05-09	$1500	$7\frac{1}{2}$%	60 d
(c)	03-15	$850	8%	90 d
(d)	10-27	$265	7%	30 d
(e)	12-10	$1250	7%	15 d
(f)	06-04	$2500	$6\frac{1}{2}$%	30 d

3 d grace!

2. Find the proceeds from the following interest-bearing notes discounted at 8%.

	Date	Face	Interest Rate	Time	Discount Date
(a)	01-11	$350	6%	30 d	01-30
(b)	03-17	$600	9%	60 d	04-03
(c)	04-26	$1500	8%	20 d	04-30
(d)	05-12	$200	10%	15 d	05-15
(e)	07-30	$920	7%	90 d	08-30
(f)	09-05	$150	12%	30 d	09-15

$Pr = A - Art$

$r = \frac{I}{Pt}$

3. Miss Crawford holds a noninterest-bearing note for $5780, payable on 03-31. On 03-15, she takes the note to the bank and the bank discounts it at 8%. What proceeds does Miss Crawford receive?

4. Consider a loan of $100 for one year discounted at 8%. Using the proceeds as principal and the discount as interest, calculate the actual rate of interest you are paying.

5. On 02-16, Mr. Taylor was unable to meet a debt of $562. He gave a 60 d note bearing interest of 10% to his creditor. On 03-01, the creditor took the note to the bank which discounted it at $7\frac{1}{2}$%. Find the discount period and the proceeds.

6. Mr. R. Pope, a plumbing contractor, was forced to pay for a shipment of pipe with a promissory note when labour trouble held up completion of a job. The note for $2560 had a time of 30 d with interest at 6%. The note was made on 06-03 and the pipe company had it discounted at the bank on 06-10 at 7%. What were the proceeds?

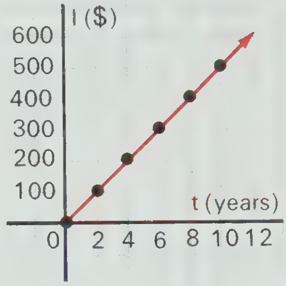
12.7 GRAPHS OF INTEREST RELATIONS $I = Prt$

When a set principal is invested at a fixed rate of interest P and r become constants in the relation $I = (Pr)t$.

Recall the graph of direct variation from Chapter 1.

$$\text{If } I = kt \qquad (k \text{ constant})$$
$$\text{then we can say } I \propto t$$
$$\text{or } I \text{ varies directly as } t$$

EXAMPLE 1. *If \$500 is invested at 10% simple interest, graph the relation between I and t for 0 to 12* a.



If $P = \$500$ and $r = 10\%$
then $I = 50t$

t	0	2	4	6	8	10	12
I	0	100	200	300	400	500	600

If the formula $I = Prt$ is solved for P

$$P = \frac{I}{rt}$$

If I and t are constant the relation becomes

Recall the graph of an inverse relation from Chapter 1.

$$P = k\frac{I}{r} \qquad (k \text{ constant})$$

$$\text{or } P \propto \frac{1}{r}$$

P varies inversely as r

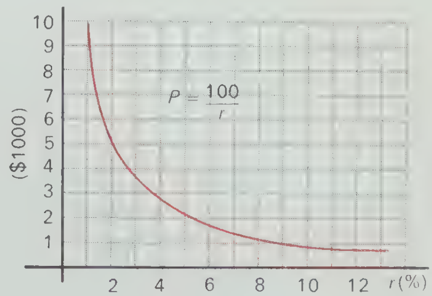
EXAMPLE 2. *If \$100 interest is to be earned in 1* a, *graph the relation between P and r.*

Solution

If $I = \$100$ and $t = 1\text{a}$

$$P = \frac{100}{r}$$

r(%)	1	2	5	10
P($)	10 000	5000	2000	1000



EXERCISE 12-7

How big a number is 10^{100}? Are there 10^{100} grains of sand on earth? Try the following question before you answer. The mass of a hydrogen atom is 1.66×10^{-24} g. The mass of the sun is 1.99×10^{33} g. The composition of the sun is mostly hydrogen. If it were all hydrogen how many atoms would there be in the sun?

A

1. Solve the equation $I = Prt$ for P, r and t.
2. Supply the missing parts of the table

	I($)	P($)	r(%)	t(a)
(a)		400	10	1
(b)		100	5	$\frac{1}{2}$
(c)	12		6	1
(d)	50		5	2
(e)	30	300		1
(f)	3	100		$\frac{1}{4}$
(g)	8	50	8	
(h)	60	1000	12	

$I = Prt$

$P = \frac{I}{rt}$

$r = \frac{I}{Pt}$

$t = \frac{I}{Pr}$

3. $I = 80t$
(a) Describe the shape of the graph.
(b) What is the slope of the graph?
(c) What is the y-intercept of the graph?
(d) What interest is paid when $t = 1, 2, 3, 5$?

4. $P = \frac{50}{r}$
(a) Describe the shape of the graph.
(b) What happens to P as r gets very large?
(c) What happens to P as r approaches zero?
(d) What principal is required if $r = 2\%, 5\%, 10\%$?

B

5. Graph the relation between interest and principal where:
(a) Rate is 12% and time is 1 a.
(b) Rate is 8% and time is $\frac{1}{2}$ a.

6. Graph the relation between interest and time where:
(a) Principal is $1000 and rate is 10%.
(b) Principal is $500 and rate is 7%.

7. Graph the relation between time and principal where:
(a) Interest is $100 and rate is 6%.
(b) Interest is $60 and rate is 5%.

8. Graph the relation between time and rate where:
(a) Interest is $100 and principal is $2000.
(b) Interest is $60 and principal is $1500.

C

9. Graph the relation between amount and time where:
(a) Principal is $100 and rate is 8%.
(b) Principal is $500 and rate is 12%.

$A = P + I$
$A = P + Prt$

REVIEW EXERCISE

B 1. Find the amount to be repaid for each of the following loans. (Do not add days of grace.)
(a) \$256 at $6\frac{1}{2}$% for 8 mo.
(b) \$1594 at 8% for 30 d.
(c) \$750 at 12% for 90 d.

2. A curb-side loan shark will lend \$9 on Monday to be repaid by \$10 on payday, Friday. What interest rate is he charging?

3. A 5% Premium Savings Account pays 5% a on the minimum monthly balance with interest being added every six months, in October and April. If your minimum balances for November–April are \$235.20, \$576.30, \$347.50, \$139.23, \$295.62, and \$456.30, how much interest will be added at the end of April?

4. A loan for \$653 is to be repaid in 12 equal instalments over the next year. The interest charged is $5\frac{1}{2}$% of the full principal for one year.
(a) How much will each payment be?
(b) Find the effective annual rate.

5. A personal bank loan with life insurance included for \$3000 may be repaid by 24 monthly payments of \$138.51 or 36 monthly payments of \$96.84.
(a) Compare the effective annual rates (including insurance costs).
(b) Compare the total amounts of the charges.

6. A motorcycle costs \$375 cash or \$75 down and \$11/mo. for 30 mo. (service charge of \$1/mo.). Find the effective annual rate being charged.

7. A loan of \$700 is to be repaid at the rate of \$100 plus accrued interest per month. Interest is being charged at $\frac{3}{4}$%/mo. Find the total interest charges and the nominal annual rate.

8. Mr. Saunders has a revolving credit account at Diamond Department Store. His previous month's balance was \$79.50, and his monthly purchases have amounted to \$276.80. Service charges are $1\frac{1}{4}$% of the previous month's balance and the minimum payment is 20% of the total of purchases and previous balance. Find the amount due this month.

9. Find the proceeds from a note for \$275 discounted for 30 d at $8\frac{1}{2}$%.

10. A note with maturity value of \$1200 is discounted for 42 d at $10\frac{1}{4}$%. Find the proceeds.

11. On 04-07, Mrs. Smidgens discounts a \$500 note at the bank at 8%. The note is a 90 d note dated 03-03, bearing interest at 6%.
(a) Find the maturity value of the note.
(b) Find the proceeds from the note.

12. A 60 d note for \$760 bearing interest at 7% is discounted 45 d before maturity at $8\frac{1}{2}$%. Find the maturity value and the proceeds.

13. Graph the relation $I = Prt$ if $P = \$100$, and $r = 12\%$, $0 \le t \le 10$ y.

14. Graph the relation $t = \frac{I}{Pr}$ if $I = \$500$ and $P = \$1000$, $1 \le t \le 10$ y.

REVIEW AND PREVIEW TO CHAPTER 13

$I = Prt$

1. Find the interest when $100 is invested at the following rates per annum.

$I = Prt$

(a) 1 a at 5% (b) 1 a at 9%
(c) 1 a at 12% (d) 1 a at 8%
(e) 1 a at $10\frac{1}{2}$% (f) 1 a at $7\frac{1}{4}$%
(g) 1 a at $16\frac{3}{4}$% (h) 1 a at $12\frac{1}{2}$%
(i) $\frac{1}{2}$a at 8% (j) $\frac{1}{4}$a at 10%
(k) $\frac{3}{4}$a at 12% (l) $1\frac{1}{2}$ a at 18%

$A = P\ (1 + rt)$

2. Find the amount if $100 is invested at the following rates per annum.

$A = P + I$
$A = P + Prt$
$A = P(1 + rt)$

(a) 1 a at 6% (b) 1 a at 8%
(c) 1 a at 9% (d) 1 a at 7%
(e) 1 a at 12% (f) $\frac{1}{2}$ a at 6%
(g) $\frac{1}{4}$ a at 8% (h) $\frac{1}{3}$ a at 7%
(i) $\frac{1}{2}$ a at 7% (j) $\frac{1}{12}$ a at 12%

3. Determine the number of time intervals for each of the following:
(a) $\frac{1}{2}$ a intervals in $3\frac{1}{2}$ a.
(b) $\frac{1}{4}$ a intervals in $10\frac{1}{2}$ a.
(c) $\frac{1}{2}$ a intervals in 5 a.
(d) $\frac{1}{3}$ a intervals in $7\frac{2}{3}$ a.
(e) 2 mo. intervals in $8\frac{1}{2}$ a.
(f) $\frac{1}{4}$ a intervals in 25 a.

4. (a) Evaluate each of the following functions in n for: $n \in \{0, 1, 2, 3\}$
(i) $y = 1 + 0.1n$ (ii) $y = (1.1)^n$
(iii) $y = 3(1 + 0.1n)$ (iv) $y = 3(1.1)^n$
(b) Evaluate each of the following functions in n for: $n \in \{0, 5, 10, 15\}$
(i) $y = 1 + 0.06n$ (ii) $y = (1.06)^n$
(iii) $y = 10(1 + 0.06n)$ (iv) $y = 10(1.06)^n$

(Use slide rule and round off to 3 figures as you go)

5. Banks and businesses usually calculate the interest at the end of a conversion period so that the graph is not a smooth curve but a series of steps. The graph in Figure 13-1 is an example of this type of step function.
(a) Prepare step graphs to show the amount of $1 invested:
(i) for 3 a at 10%, compounded semi-annually.
(ii) for 1 a at 12%, compounded monthly.
(b) Investigate interest rates and methods used to calculate interest on deposits and loans of businesses in your school area.

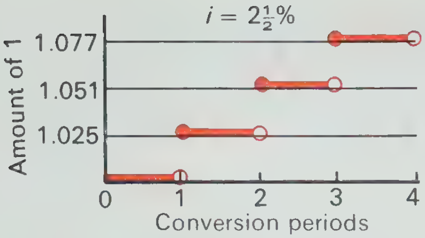


Figure 13-1

Evaluate:

1. $3\frac{1}{2}\%$ of \$122.28
2. $33\frac{1}{3}\%$ of \$474.80
3. 25% of \$437.25
4. $12\frac{1}{2}\%$ of \$643.50
5. $16\frac{2}{3}\%$ of \$1257.80
6. $22\frac{1}{2}\%$ of \$4750.18
7. 17% of \$3375.95
8. $8\frac{3}{4}\%$ of \$84 900.00
9. $9\frac{1}{4}\%$ of \$2487.49
10. $12\frac{1}{2}\%$ of \$1649.99

CHAPTER 13

Compound Interest

13.1 GROWTH OF MONEY UNDER COMPOUND INTEREST

When interest is paid or added to the principal of a loan at regular intervals during the time of the loan, the loan is said to earn compound interest.

EXAMPLE 1. *\$100 is invested at 12%/a, interest paid semi-annually. Compute the income derived from the investment.*

Solution $I = Prt$

$I = \$100 \times 0.12 \times \frac{1}{2}$

$I = \$100 \times 0.06$ (The rate per conversion paid is 6%)

$= \$6.00$

\$6 will be paid to the investor every six months.

What happens to the principal over the term of the loan?
What happens to the interest earned each period over the term of the loan?
Since the interest is paid every six months, the principal and the interest paid each conversion period will remain unchanged.

EXAMPLE 2. *\$100 is invested at 12%/a, compounded semi-annually. If the interest is added to the principal, calculate the interest and amount at the end of the first 4 conversion periods.*
(Note the amount at the end of one conversion period becomes the principal for the next.)

Solution Since the interest is compounded semi-annually, the conversion period is $\frac{1}{2}$ a.

12% compounded semi-annually = 6% each period

I = Prt A = P + I	
1. After 1 conversion period $I = \$100(0.12)(\frac{1}{2})$ $= \$100(0.06)$ $= \$6$ $A = \$100 + \$6$ $= \$106$	2. After 2 conversion periods $I = \$106(0.12)(\frac{1}{2})$ $= \$106(0.06)$ $= \$6.36$ $A = \$106 + \$6.36$ $= \$112.36$
Note in each case rate per conversion period is 0.06	
3. After 3 conversion periods $I = \$112.36(0.06)$ $= \$6.74$ $A = \$119.10$	4. After 4 conversion periods $I = \$119.10(0.06)$ $= \$7.15$ $A = \$126.25$

What happens to the principal over the term of the investment?
What happens to the interest over the term of the investment?

Since the interest is added to the principal each period, the principal and the interest earned each period increase.

INVESTIGATION 13.1

1. Complete the following table in your notebook.

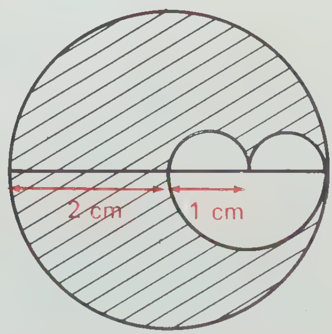
Yearly interest rate j	6%	7%	$8\frac{1}{2}$%		12%	$6\frac{1}{4}$%	10%	$8\frac{1}{4}$%
Number of conversion periods per year m	2		1	4	12	2		
Interest rate for one conversion period i	3%	$3\frac{1}{2}$%		3%			10%	$4\frac{3}{8}$%

2. Complete the following calculations to find the total amount after 2 a if \$1000 is invested at 9% and the interest is compounded semi-annually (twice a year).

Principal, P. \$1000.00
9% interest on \$1000, first half, first year $1000 \times 0.045 =$
New principal for second half of first year . . . .
Interest for second half of first year . . . .
New principal for first half of second year . . . .
Interest for first half of second year . . . .
New principal for second half of second year . . . .
Interest for second half of second year . . . .
Amount at end of second year . . . .

3. Complete the following table to find the total amount after 20 a if \$100 is invested at 8% and compounded semi-annually. Leave your answer in an unsimplified form.

Conversion period	Principal P, at beginning of conversion period	Amount A, at end of given conversion period
1	100	$100 + 100(0.04) = 100(1 + 0.04)$
2	$100(1 + 0.04)$	$100(1 + 0.04)(1 + 0.04) = 100(1.04)^2$
3	$100(1 + 0.04)^2$	
4		
5		
6		
...		
19		
20		



Can you find the area of the shaded portion?

4. (a) Repeat question 3 to find the total amount A after n conversion periods, if a principal P is invested at rate i for one conversion period.
(b) State a formula to find the amount A, given principal P, interest rate

per conversion period i, and the total number of conversion periods n.
(c) Test your formula stated in (b) using the information in question 2.

> The compound amount A of a principal P for n conversion periods is given by
>
> $$A = P(1 + i)^n$$
>
> for the interest rate i per conversion period.

5. Write the following in the form $P(1 + i)^n$.
(a) \$1000 at 9%/a, compounded annually for 10 a.
(b) \$325 at 12%/a, compounded monthly for 36 mo.
(c) \$117 at 10%/a, compounded semi-annually for 18 mo.
(d) \$200 at $11\frac{1}{4}$%/a, compounded semi-annually for $3\frac{1}{2}$ a.
(e) \$5000 at 8%/a, compounded quarterly for $5\frac{1}{2}$ a.
(f) \$4500 at 9%/a, compounded semi-annually for $9\frac{1}{2}$ a.
(g) \$600 at 18%/a, compounded monthly for $2\frac{1}{4}$ a.
(h) \$2500 at 12%/a, compounded bi-monthly for $4\frac{1}{2}$ a.

EXAMPLE 3. *Find the amount of \$2000 at 12%/a compounded semi-annually for 3* a.

$$\begin{array}{r} 1.06 \\ 1.06\ \times \\ \hline 1.1236 \\ 1.06\ \times \\ \hline 1.191016 \\ 1.06\ \times \\ \hline 1.262476\ldots \\ 1.06\ \times \\ \hline 1.338225\ldots \\ 1.06\ \times \\ \hline 1.418519\ldots \end{array}$$

Solution

$$\begin{aligned} P &= \$2000 \\ i &= \frac{12\%}{2} \\ &= 6\% \\ &= 0.06 \\ n &= 3 \times 2 \\ &= 6 \end{aligned}$$

$$\begin{aligned} A &= P(1 + i)^n \\ A &= \$2000(1.06)^6 \\ &= \$2000(1.418519) \\ &= \$2837.04 \end{aligned}$$

EXERCISE 13-1

A 1. Give the rate (i) per conversion period

	Rate per annum (i)	Compounded		Rate per annum (i)	Compounded
(a)	10%	Semi-annually	(b)	12%	Annually
(c)	8%	Semi-annually	(d)	7%	Semi-annually
(e)	12%	Quarterly	(f)	9%	Semi-annually
(g)	8%	Quarterly	(h)	15%	Quarterly
(i)	12%	Monthly	(j)	18%	Monthly

B 2. Find the income derived from each of the following investments for each conversion period.
(a) \$5000 invested at 8% compounded semi-annually.
(b) \$8000 invested at 12% compounded quarterly.
(c) \$7500 invested at 10% compounded semi-annually.
(d) \$500 invested at 12% compounded monthly.
(e) \$12 000 invested at 18% compounded bi-monthly.

Unless otherwise stated the interest rate given is per annum.

(f) $4500 invested at 9% compounded semi-annually.

3. $500 is invested at 12%/a, compounded semi-annually. The interest is added to the principal at the end of each conversion period. Calculate the interest and the amount at the end of the first 4 conversion periods.

4. $100 is borrowed at 12%, compounded quarterly. The interest is added to the principal. Calculate the amount required to repay the loan after 15 mo.

Compare
$(4^3)^2$ and $4^{(3^2)}$
What is the largest number you can make up using 3, 4, and 5?

> Unless stated otherwise it will be assumed that the interest is added to the principal at the end of each conversion period.

5. (a) Find the amount when $5000 is invested at 10%/a, compounded semi-annually for 3 a.
(b) Find the amount when $5000 is invested at 10%/a, simple interest for 3 a.

6. Find the amount when $2500 is invested at 18%/a, compounded quarterly for 2 a.

7. Find the amount required to repay a debt of $300 when interest is charged for $1\frac{1}{2}$ a at 8%/a, compounded semi-annually.

13.2 COMPOUND INTEREST TABLES

A = amount
i = interest rate per conversion period
P = principal
n = number of conversion periods

To reduce the work required to find the amount of an investment, tables of values for $(1 + i)^n$ have been calculated for various values for i and n.

Since: $A = P(1 + i)^n$

to find the amount for a given investment it is only necessary to multiply the appropriate value in the table by the principal invested.

i / n	4%	5%
1	1.040 00	1.050 00
2	1.081 60	1.102 50
3	1.124 86	1.157 63
4	1.169 86	1.215 51
5	1.216 65	1.276 28
6	1.265 32	1.340 10
7	1.315 93	1.407 10
8	1.368 57	1.477 46
9	1.423 31	1.551 33
10	1.480 24	1.628 89
11	1.539 45	1.710 34
12	1.601 03	1.795 86
13	1.665 07	1.885 65
14	1.731 68	1.979 93
15	1.800 94	2.078 93
16	1.872 98	2.182 87
17	1.947 90	2.292 02
18	2.025 82	2.406 62
19	2.106 85	2.526 95
20	2.191 12	2.653 30
21	2.278 77	2.785 96
22	2.369 92	2.925 26
23	2.464 72	3.071 52
24	2.563 30	3.225 10
25	2.665 84	3.386 35

EXAMPLE 1. *Find the amount of an investment of $1000 invested at 10%/a, compounded semi-annually for 12 a.*

Solution

$A = P(1 + i)^n$

$P = \$1000$ $\qquad i = \frac{1}{2}(10\%)$ (compounded semi-annually)

$= 5\%$

$n = 12 \times 2$ (compounded semi-annually)

$= 24$

$A = \$1000(1.05)^{24}$

$= \$1000(3.22510)$ (from table)

$= \$3225.10$

The amount is $3225.10.

EXERCISE 13-2

A

1. Find i and n for each of the following:
(a) 12% compounded semi-annually for 6 a.

(b) 18% compounded quarterly for 8 a.
(c) 8% compounded annually for 10 a.
(d) 7% compounded semi-annually for 5 a.
(e) 12% compounded monthly for 3 a.
(f) 11% compounded semi-annually for 9 a.
(g) 10% compounded quarterly for $4\frac{1}{2}$ a.
(h) 9% compounded bi-monthly for $5\frac{1}{2}$ a.

2. Read from the tables:

(a) $(1.015)^4$	(b) $(1.04)^{12}$	(c) $(1.08)^{20}$
(d) $(1.035)^{18}$	(e) $(1.05)^{35}$	(f) $(1.06)^{24}$
(g) $(1.03)^{11}$	(h) $(1.055)^{33}$	(i) $(1.07)^{16}$
(j) $(1.025)^{36}$	(k) $(1.005)^{25}$	(l) $(1.045)^{14}$
(m) $(1.07)^{20}$	(n) $(1.03)^{11}$	(o) $(1.08)^{10}$

B **3.** Find the amount of the following investments:

	Principal	Annual rate	Conversion period	Term
(a)	$100	8%	annual	4 a
(b)	$1000	9%	semi-annual	$6\frac{1}{2}$ a
(c)	$100	12%	quarterly	5 a
(d)	$10 000	6%	annual	10 a
(e)	$100	10%	quarterly	33 mo.
(f)	$100	11%	semi-annual	20 a
(g)	$100 000	9%	semi-annual	18 mo.
(h)	$1 000 000	12%	quarterly	3 mo.

4. Use the tables to find the following amounts:

(a) $100(1.025)^{14}$	(b) $25(1.04)^{8}$
(c) $500(1.07)^{12}$	(d) $500(1.035)^{15}$
(e) $150(1.025)^{15}$	(f) $1000(1.015)^{24}$
(g) $450(1.07)^{12}$	(h) $1500(1.08)^{25}$
(i) $700(1.055)^{30}$	(j) $500(1.045)^{16}$

5. Find the amount of the following:
(a) $700 invested at 8% for 2 a, compounded annually.
(b) $125 invested at 7% for 2 a, compounded annually.
(c) $600 invested at 12% for 3 a, compounded quarterly.
(d) $5000 invested at 7% for 12 a, compounded annually.
(e) $5000 invested at 7% for 12 a, compounded semi-annually.
(f) $3000 invested at 10% for 4 a, compounded quarterly.
(g) $12 000 invested at 11% for 15 a, compounded semi-annually.
(h) $25 000 invested at 9% for 20 a, compounded semi-annually.

What value must $(1 + i)^n$ have to double the principal?

6. (a) How long must money be invested at 8% compounded annually to double? (To the nearest year.)
(b) How long must money be invested at 8% compounded semi-annually to double? (To the nearest half year.)
(c) How long must money be invested at 8% compounded quarterly to double? (To the nearest quarter year.)

7. In order to double an investment in 10 a, at what rate compounded annually must the money be invested? (To the nearest percent.)

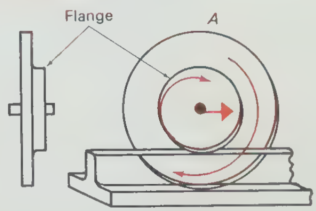
8. When Joe Smith was born his grand-father invested \$100 in Joe's name at 10%, compounded semi-annually.
(a) If Joe cashed in the investment when he was 19 a old how much did he get?
(b) If Joe waited until he was 20 a old how much did he get?
(c) How much did he gain by waiting the extra year?

C **9.** Complete the table to show the amount of \$1 at the end of a number of years, invested at 7% and 8% interest, compounded annually.

End of year	Amount at 7%	Amount at 8%
1		
2		
3		
4		
5		
10		
20		
30		
40		

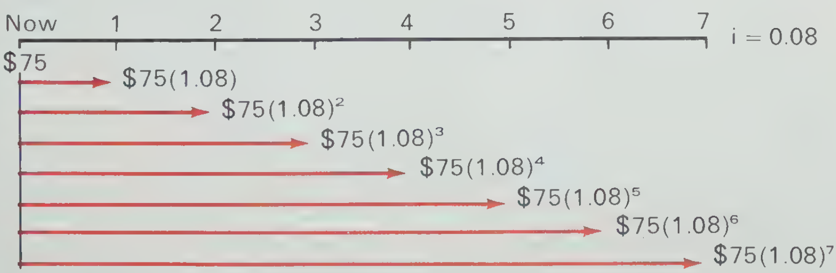
(a) What is the amount when \$1000 is invested at 7% compounded annually for:
(i) 5 a?
(ii) 10 a?
(b) What is the amount when \$1000 is invested at 8% compounded annually for the same periods?
(c) Construct a graph showing how much \$1 would amount to, invested at 7% and compounded annually from 1 to 10 a. Show on the same graph what \$1 would amount to invested at 8%, compounded annually.

As the flanged wheel rolls along the rail in which direction is the bottom of the wheel moving?
Can you trace the path of point *A* on the wheel?
How does a train move forward if the bottoms of the wheels are moving backward?



13.3 PROBLEMS INVOLVING AMOUNT

The growth of money, period by period, can be illustrated on a time diagram. The following time diagram shows the amount of \$75 at 8%, compounded annually for any number of interest periods from 1 to 7.



The amount $\$75(1.08)^5$ is found by using the tables in the appendix to find the accumulation factor $(1.08)^5$.

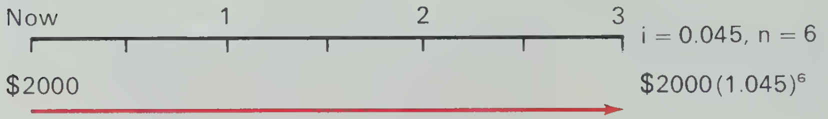
$$(1.08)^5 = 1.469\,33$$
$$\$75(1.08)^5 = \$75(1.469\,33)$$

$= \$110.199$

Hence the compound amount is \$110.20 to the nearest cent.

EXAMPLE 1. *Find to the nearest cent the compound amount of \$2000 invested at 9%/a, compounded semi-annually for 3* a.

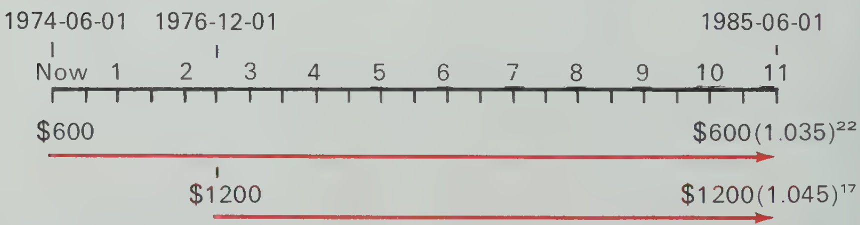
Solution 9%/a, compounded semi-annually:



$$\begin{aligned} A &= \$2000(1.045)^6 && \text{from the diagram} \\ &= \$2000 \times 1.302\,26 && \text{from the table} \\ &= \$2604.52 \end{aligned}$$

Hence the compound amount is \$2604.52 to the nearest cent.

EXAMPLE 2. *If Miss Samual invested \$600 on 1974-06-01 at 7% compounded semi-annually, and invests an additional \$1200 on 1976-12-01 at 9% compounded semi-annually, how much is her investment worth on 1985-06-01?*



Solution

$$\begin{aligned} A &= \$600(1.035)^{22} + \$1200(1.045)^{17} \\ &= \$600(2.131\,51) + \$1200(2.113\,38) \\ &= \$1278.91 + \$2536.06 \\ &= \$3814.97 \end{aligned}$$

The total amount of the investment is \$3814.97.

EXERCISE 13-3

B

1. Make time diagrams to show the compound amount for each of the following. Calculate the amount.

(a) \$400 in 5 a at 8%, compounded annually.
(b) \$550 in 2 a at 7%/a, compounded semi-annually.
(c) \$75 in 18 mo. at 12%/a, compounded monthly.
(d) \$1200 in 2 a at $3\frac{1}{2}$% per half-year, compounded semi-annually.
(e) \$100 in 12 mo. at 1%/mo., compounded monthly.

2. A man borrows \$2000 from the bank at 7%/a, compounded semi-annually. Find the amount he must pay in 36 mo.

3. A man leaves his 14-year-old granddaughter \$25 000 invested at 8%/a, compounded semi-annually until her twenty-first birthday. How much does she receive on that birthday? (Assume it was invested on the girl's fourteenth birthday.)

4. An aunt leaves \$100 000 to be divided equally between her two nephews, who are 13 and 15 a old. Instead of giving each \$50 000, the money is invested at 9%/a, compounded semi-annually. How much does each receive at the age of 18?

5. (a) Find the compound amount to the nearest 10¢ at the end of 18 a if a principal of \$1000 is invested over the first half of the term at 8%/a, compounded semi-annually, and over the second half of the term at 11%/a, compounded semi-annually.
(b) Find the compound amount to the nearest 10¢ if the principal of \$1000 had been invested at 10%/a, compounded semi-annually over the 18-a term.

6. On her son's first birthday a woman invests \$1000 at 8%/a, compounded semi-annually, to provide for the boy's education. How much money will the boy have on his eighteenth birthday?

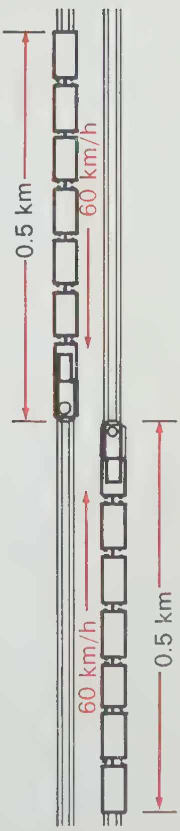
7. On the birth of his son, a man invests \$500 at 7%/a, compounded semi-annually, to provide for the boy's education. If he invests \$500 again on each of the boy's first, second and third birthdays, how much will the boy have at the age of 18? Begin with a time diagram.

8. A bank pays 8%/a, compounded semi-annually, on its deposits, and adds interest to a depositor's account on 06-01 and 12-01 of each a. On 1975-06-01 P. J. Jones deposited \$1200. If no withdrawals were made, how much was on deposit on 1977-12-01.

9. A financial concern which pays 9%/a, compounded semi-annually, on the minimum half-yearly balance adds interest to deposits on 01-01 and 07-01 of each year. T. J. Smith deposited \$500 on 1977-07-01 and \$300 on 1977-12-29. How much did he have on deposit on 1979-01-01 if he made no withdrawals?

10. Mrs. D. B. Campbell opened a savings account on 02-01 and deposited \$500; 03-01, deposited \$300; 04-01, withdrew \$200.
If interest is credited 06-01 and 12-01 at the annual rate of 5%, compounded semi-annually, on the minimum balance for the preceding six months, how much is credited by the bank to Mrs. Campbell's account on 12-01?

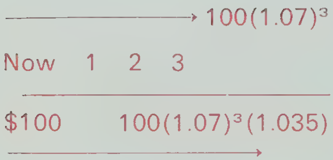
11. If \$150 is deposited in an account every three months for 3 a (12 deposits starting now), what is the value of the investment at the time of the last deposit? Interest is paid at the rate of 8%, compounded quarterly.



Two trains each 0.5 km long and traveling at 60 km/h pass each other on adjacent tracks. How long is it from when the engines meet to when the cabooses part?

13.4 FRACTIONAL PARTS OF A TIME PERIOD

The formula $A = P(1 + i)^n$ was derived on the assumption that n was a whole number and that the money would be invested for a whole number of conversion periods. Normally when you close an account at the bank, interest is paid up to the end of the last conversion period. There may be occasions when it is convenient to calculate the amount for a fractional part of a period. When computing compound interest for a fraction of a conversion period, simple interest is used.



EXAMPLE 1. *Find the amount of \$100 in $3\frac{1}{2}$ a at 7%/a compounded annually.*

Solution

1. For three years

$A = P(1 + i)^n$ $\quad P = \$100 \quad i = 0.07 \quad n = 3$

$A = \$100(1.07)^3$

$\quad = \$100(1.225\,04)$ $\quad$ (from table)

$\quad = \$122.50$

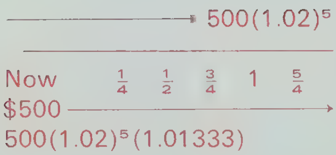
2. For the last $\frac{1}{2}$ a

$A = P(1 + rt) \quad P = \$122.50 \quad r = 0.07 \quad t = \frac{1}{2} \quad rt = 0.035$

$A = \$122.50(1.035)$

$\quad = \$126.79$

The amount is \$126.79.



EXAMPLE 2. *Find the amount of \$500 invested for 17 mo. at 8%/a, compounded quarterly.*

Solution

For the first 15 mo. (5 quarters)

$A = P(1 + i)^n \quad P = \$500 \quad i = 0.02 \quad n = 5$

$\quad = \$500(1.02)^5$

$\quad = \$500(1.104\,08)$

$\quad = \$552.04$

For the last 2 mo.

$A = P(1 + rt) \quad P = \$552.04 \quad r = 0.08 \quad t = 0.1666$

$rt = 0.013\,33$

$\quad = \$552.04(1.013\,33)$

$\quad = \$559.40$

The amount of the investment is \$559.40.

Quarterly means every 3 mo.

EXERCISE 13-4

1. Calculate the amount in each of the following. Make a time diagram.
(a) \$3000 for $5\frac{1}{2}$ a at 7%/a, compounded annually.
(b) \$125 for 9 mo. at 10%/a, compounded semi-annually.
(c) \$2500 for $3\frac{1}{4}$ a at 8%/a, compounded semi-annually.
(d) \$125 for 7 mo. 10 d at 1%/mo., compounded monthly.

2. Calculate the amount in each of the following:
(a) \$800 for $6\frac{1}{2}$ a at 7%/a, compounded annually.
(b) \$1000 for $3\frac{3}{4}$ a at 12%/a, compounded semi-annually.
(c) \$125 for 11 mo. at 8%/a, compounded quarterly.
(d) \$250 for 9 mo. 12 d at 10%/a, compounded quarterly.

3. Find the amount when \$750 is invested at 9%/a, compounded semi-annually for 5 a, 83 d.

4. \$500 is borrowed for 3 mo., 15 d. If interest is charged at 1%/mo., compounded monthly, what is the amount of the loan?

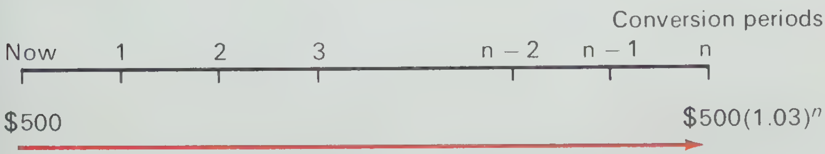
5. \$800 is borrowed for 2 a, 3 mo., 12 d. Interest is charged at $1\frac{1}{2}$%/mo., compounded monthly. Find the total interest charged.

13.5 PROBLEMS INVOLVING RATE AND TIME

How long will it take a sum of money to double at 6%/a, compounded semi-annually? What interest rate compounded annually must be charged to triple a sum of money in 20 a? The answers to these questions are helpful in planning long-term investments.

EXAMPLE 1. *In how many years will \$500 double if it is loaned at 6%/a, compounded semi-annually?*

Solution 6%/a compounded semi-annually gives $i = 0.03$.



$$500(1.03)^n = 1000 \qquad \text{(from time diagram)}$$
$$(1.03)^n = 2$$
$$n \doteq 24 \qquad \text{(from the table)}$$

The money will double after approximately 24 conversion periods (12 a).

n \ i	3%
1	1.030 00
2	1.060 90
3	1.092 73
4	1.125 51
5	1.159 27
6	1.194 05
7	1.229 87
8	1.266 77
9	1.304 77
10	1.343 92
11	1.384 23
12	1.425 76
13	1.468 53
14	1.512 59
15	1.557 97
16	1.604 71
17	1.652 85
18	1.702 43
19	1.753 51
20	1.806 11
21	1.860 29
22	1.916 10
23	1.973 59
24	2.032 79
25	2.093 78

The correct answer to the example is slightly less than 12 a. A better approximation can be found by interpolating from the tables.

$$\left.\begin{array}{l}(1.03)^{23} = 1.973\ 59\\ \\(1.03)^{24} = 2.032\ 79\end{array}\right\}0.059\ 20 \qquad \left.\begin{array}{l}(1.03)^{23} = 1.973\ 59\\ \\(1.03)^{n} = 2.000\ 00\end{array}\right\}0.026\ 41$$

The difference 0.5920 corresponds to six months. The difference 0.026 41 corresponds to $\dfrac{0.026\ 41}{0.059\ 20} \times 6 \doteq 2.7$ mo. Twenty-three conversion periods represent $11\frac{1}{2}$ a or 11 a, 6 mo. Therefore the money will double in 11 a, 6 mo., plus 3 mo., or 11 a, 9 mo. (to the nearest month).

EXAMPLE 2. *What annual rate of interest to the nearest $\frac{1}{10}$% must be charged for \$1000 to double in 15 a if interest is compounded semi-annually?*

Solution

From the tables,

$$\left.\begin{array}{l}(1+0.02)^{30} = 1.811\ 36\\ \\(1+0.025)^{30} = 2.097\ 57\end{array}\right\}0.286\ 21 \qquad \left.\begin{array}{l}(1+0.02)^{30} = 1.811\ 36\\ \\(1+i)^{30} = 2.000\ 00\end{array}\right\}0.1886$$

The difference 0.286 21 corresponds to $\frac{1}{2}$%.

The difference 0.188 64 corresponds to $\dfrac{0.189}{0.286} \times 0.500\%$

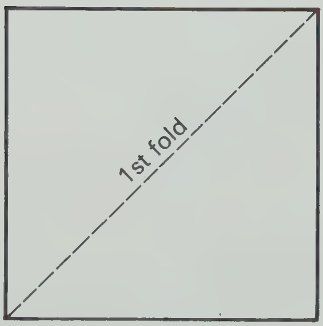
$= 0.331\%$ (3 significant figures)

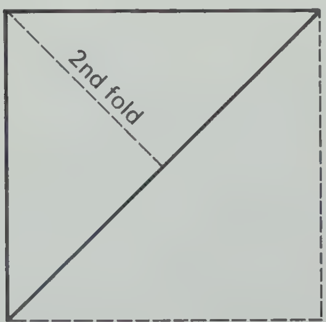
$i = 2\% + 0.331\%$

$= 2.331\%$

$2i = 4.662\%$

$\therefore$ The annual interest rate is 4.7% to the nearest $\frac{1}{10}$%.





EXERCISE 13-5

A

1. Find from the tables the value of n to the nearest whole number.
   (a) $(1.015)^n = 1.200$ (b) $(1.03)^n = 1.800$ (c) $(1.05)^n = 3$
   (d) $(1.08)^n = 5$ (e) $(1.045)^n = 4$ (f) $(1.005)^n = 1.2$

B

2. Find from the tables the value of i to the nearest $\frac{1}{10}$%.
   (a) $(1+i)^{12} = 1.2$ (b) $(1+i)^{32} = 3$ (c) $(1+i)^{7} = 1.5$
   (d) $(1+i)^{22} = 3$ (e) $(1+i)^{11} = 1.5$ (f) $(1+i)^{18} = 3.75$

3. In how many years will \$500 double at 5%/a, compounded semi-annually?

4. In how many years will \$100 triple if it is invested at 9%/a compounded semi-annually?

5. In how many years and months will \$100 double if it is invested at 8%/a, compounded semi-annually?

6. What annual rate must be charged to double \$200 in 18 a if the interest is compounded annually?

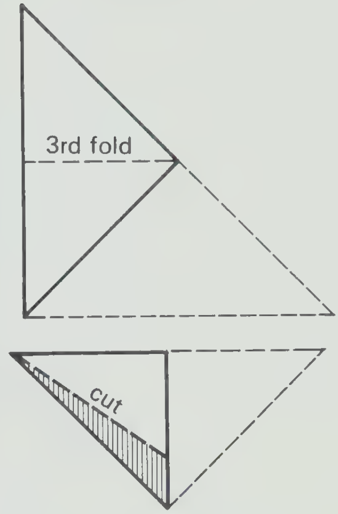
7. What annual rate to the nearest $\frac{1}{2}$% must be charged to triple a sum of money in 15 a, if interest is compounded semi-annually?

8. What annual rate compounded semi-annually must be charged to increase \$1000 to \$2500 in 10 a?

C 9. What principal invested at 12% compounded semi-annually for 5 a gives the same amount as \$400 invested for the same period at 8% compounded semi-annually?

10. How long must \$600 be invested at 9% compounded semi-annually to give the same return as if it had been invested for 7 a at 6% compounded semi-annually?

11. At what rate must \$500 be invested for 6 a to give the same return as \$500 invested for 4 a at 8% compounded annually?



Can you visualize what shape the shaded portion will fold out to?

13.6 PRESENT VALUE OF AN AMOUNT

The principal, P, which must be invested now at a set rate of interest to produce a given amount, A, after a specified time is called the present value. In Investigation 13.1 we established the formula

$$A = P(1 + i)^n$$

Solving for P, we have $P = \dfrac{A}{(1 + i)^n}$ or

$$P = A(1 + i)^{-n}$$

The values of $(1 + i)^{-n}$ can be found in the table in the appendix.

EXAMPLE 1. *Find the present value of \$1000 due in 10 a at 5%/a, compounded annually.*

Solution Let the present value in dollars be P. Then

$$\begin{aligned} P &= \$1000(1.05)^{-10} \\ &= \$1000 \times 0.613\,91 \\ &= \$613.91 \end{aligned}$$

The present value of \$1000 in 10 a at 5% is \$613.91.

EXAMPLE 2. *On the birth of a child, what sum of money should a family invest at 9%/a, compounded semi-annually, to provide the child with \$1000 on his sixteenth birthday?*

Solution 9%/a, compounded semi-annually for 16 a
$i = 0.045 \qquad n = 32$

$$P = A(1 + i)^{-n}$$
$$= \$1000(1.045)^{-32}$$
$$= \$1000 \times 0.244\ 50$$
$$= \$244.50$$

The family should invest \$244.50.

To compare the value of investments made at different times compare their present values.

EXAMPLE 3. *Compare the following two investments today.*
i) *\$1000 due in 2* a *at 13%/*a*, compounded semi-annually.*
ii) *\$1200 due in 6* a *at 11%/*a*, compounded annually.*

Solution
i) Present value of \$1000 due in 2 a where

$i = 0.065$, $n = 4$

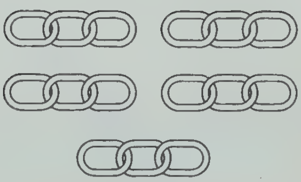
$$P = A(1 + i)^{-n}$$
$$= \$1000(1.065)^{-4}$$
$$= \$1000 \times 0.777\ 32$$
$$= \$777.32$$

ii) Present value of \$1200 due in 6 a where

$i = 0.11$, $n = 6$

$$P = A(1 + i)^{-n}$$
$$= \$1200(1.11)^{-6}$$
$$= \$1200 \times 0.534\ 64$$
$$= \$641.57$$

Hence \$1000 due in two years at 13%, compounded semi-annually, is worth more today than \$1200 due in six years at 11%/a, compounded annually.



If it costs 25¢ to break and rejoin a link, what is the least expensive way to make one chain.

EXERCISE 13-6

B

1. Find the present value of each of the following. Begin with a time diagram.
(a) \$4000 due in 6 a at 8%/a, compounded semi-annually.
(b) \$180 due in 2 a at 12%/a, compounded quarterly.
(c) \$180 due in 2 a at 12%/a, compounded monthly.
(d) \$5000 due in 10 a at 8%/a, compounded quarterly.

2. (a) What sum of money should a family invest on the birth of a child to provide \$1000 at the age of 18 if the money is to be invested at 9%, compounded semi-annually.

(b) How much should a family invest at the birth of a child to provide \$500 at the age of 18 and \$1000 at the age of 20 if money is worth 8%/a, compounded semi-annually?

(c) If bank interest is 9%/a, compounded semi-annually, what sum should you deposit today to have \$15 000 in 10 a?

(d) If money is worth 8% compounded quarterly, what sum must be invested now to purchase an automobile worth \$4650 in four years?

3. What sum of money paid today is equivalent to payments of \$1000 a year from now and \$1000 two years from now, if money bears interest at 7%/a, compounded semi-annually?

4. Which is worth more today, \$100 due in 4 a at 8%/a, compounded annually, or \$125 due in 8 a at 12%/a, compounded semi-annually? By how much?

5. A man owes \$1000 due in 3 a and \$500 due in 5 a. How much could he pay now to settle these debts if money is worth 11%/a, compounded semi-annually?

6. A man holds investments in the amounts of \$3000 due in 3 a at 7%/a, compounded semi-annually; \$5000 in 7 a at 9%/a, compounded semi-annually; and \$4000 in 5 a at 8%/a, compounded semi-annually.

(a) How much money does he have today?

(b) If he were to reinvest the three investments at 10%/a, compounded semi-annually, how much would he have in:

(i) 2 a? (ii) 5 a?

7. How much money must be invested at 9% compounded semi-annually to give an amount of \$6000 in 5 a?

8. (a) Which has the greater present value:

(i) An investment at 8%, compounded annually, worth \$3000 in 7 a.

(ii) An investment at 9%, compounded semi-annually, worth \$5000 in 10 a.

(b) By how much is it greater?

C **9.** A student will require \$2000/a for three years for further education. What amount must be invested now at 10%, compounded semi-annually, if the first payment will be required in four years.

(Hint: Find the present value of three amounts of \$2000 for terms of 4, 5, and 6 a.)

10. Find the present value of 4 payments of \$300 made at 6 mo. intervals. The first payment will be made $2\frac{1}{2}$ a hence. Money is worth 8% compounded semi-annually.

11. (a) William Saxon received a legacy worth \$1000 on his 21st birthday. If money is worth 9% compounded semi-annually, what would the legacy be worth on his 18th birthday?

(b) How much will the legacy be worth if he leaves it invested at the same rate until his 30th birthday?

12. What sum of money paid in 5 a is equivalent to $3000 paid in one year if money is worth 12%/a, compounded semi-annually?

REVIEW EXERCISE

1. Using the tables in the appendix, prepare a table of values and sketch the following on graph paper:

(a) Graph A

(i) $A = (1.05)^n, 0 \leqq n \leqq 20$

(ii) $A = (1.04)^n, 0 \leqq n \leqq 20$

(iii) $A = (1)^n, 0 \leqq n \leqq 20$

(b) Graph B

(i) $A = (1.025)^n, 0 \leqq n \leqq 40$

(ii) $A = (1.02)^n, 0 \leqq n \leqq 40$

(iii) $A = (1)^n, 0 \leqq n \leqq 40$

(c) Use the above graphs to find the amounts of the following:

(i) $50 due in 3 a at 4%/a, compounded annually.

(ii) $100 due in 15 a at 4%/a, compounded semi-annually.

(d) Use Graph B to find the present value of $750 due in 10 a at 5%/a, compounded semi-annually.

2. Find the compound amount of the following:

(a) $3500 at 7% for 3 a, compounded semi-annually.

(b) $100 at 1%/mo., compounded monthly for 10 mo.

(c) $1200 at 10%/a, compounded semi-annually for 18 mo.

(d) $2400 at 7%/a, compounded semi-annually for 36 mo.

(e) $4500 at 9%/a, for 12 a compounded semi-annually.

(f) $900 at $1\frac{1}{2}$%/mo., compounded monthly for 30 mo.

3. Find the present value of the following:

(a) $1000 due in 3 a at 7%/a, compounded semi-annually.

(b) $3000 due in 2 a at 5%/a, compounded semi-annually.

(c) $1200 due in 2 a at 8%/a, compounded semi-annually.

(d) $1200 due in 2 a at 8%/a, compounded quarterly.

(e) $500 due in 24 mo. at 1%/mo., compounded monthly.

(f) $100/mo. paid for 3 mo., the first payment made 6 mo. from now. Money is worth $1\frac{1}{2}$%/mo. compounded monthly.

4. Find the compound amount of the following:

(a) $500 for $3\frac{1}{2}$ a at 5%/a, compounded annually.

(b) $100 for $2\frac{1}{4}$ a at 6%/a, compounded semi-annually.

(c) $1500 for 4 a, 76 d at 8%/a, compounded annually.

(d) $325 for 46 mo. at 12%/a, compounded quarterly.

(e) $600 for 300 d at 8%/a, compounded quarterly.

300 d = 0.8219 a (from table). After 0.75 a how much is left of 0.8219 a?

(f) $750 for 500 d at 9%/a, compounded semi-annually.

5. In how many years will $100 double if it is invested at 7%/a, compounded semi-annually?

6. What is the smallest rate of interest to the nearest $\frac{1}{10}$% that must be charged for $650 to triple in 18 a, if the interest is to be compounded semi-annually?

7. What sum of money should a family invest on the birth of a child to provide \$600 at the age of 18 and \$1000 at the age of 20? The money is to be invested at 7%/a, compounded semi-annually.

8. In how many years will \$500 double if it is invested at 8%/a, compounded quarterly?

9. Which has the greater present value and by how much?
(a) \$600 due in 5 a at 8%/a, compounded quarterly.
(b) \$600 due in 6 a at 7%/a, compounded annually.

C **10.** What sum of money paid 5 a from today is equivalent to payments of \$1500, 2 a from now and \$3500 paid 7 a from now, if money is worth 8%/a, compounded semi-annually?

11. Find the present value of 5 annual payments of \$500 if the first payment is made 1 a from now and money is worth 8% compounded semi-annually.

12. Find the present value of 6 semi-annual payments of \$100 if the first payment is made 2 a from now and money is worth 9% compounded semi-annually.

13. \$50/mo. is placed in an account for 6 mo. Interest at 1%/mo. is added to the account at the end of each month after the first deposit.
(a) How much is in the account at the time of the last deposit?
(b) One month after the last deposit, withdrawals of \$50/mo. are made for 6 mo. How much is in the account after the last withdrawal?

A game for two players
Make 11 marks on a piece of paper / / / / / / / / / / /
Each player in turn may stroke off 1, 2 or 3 marks.
X X X / / / / / / / /. The player who stokes the last mark wins.
Devise a strategy so the first player always wins.

REVIEW OF BASIC CALCULATING SKILLS

1. Addition

(a)	(b)	(c)	(d)
$ 25.36	$199.84	$823.65	$3429.15
471.59	364.17	251.43	517.34
8.75	801.42	562.14	842.75
97.31	83.20	547.28	84.27

(e)	(f)	(g)	(h)
$279.21	$873.62	$647.52	$721.16
610.44	84.13	485.29	9.43
934.82	156.21	146.24	17.50
106.39	273.94	735.91	479.32

(i)	(j)
$5295.60	$410.23
1029.36	528.16
3500.14	476.04
721.18	859.36

2. Subtraction

(a)	(b)	(c)	(d)
$784.63	$645.83	$532.71	$1756.43
281.42	423.51	311.50	893.52

(e)	(f)	(g)	(h)
$824.59	$239.84	$573.24	$746.15
643.75	178.15	185.12	538.42

(i)	(j)
$284.15	$943.20
97.86	215.12

3. Multiplication (answers to the nearest cent).

(a)	(b)	(c)	(d)
$37.24	$57.34	$29.14	$152.84
1.07	0.065	0.93	0.0825

(e)	(f)	(g)	(h)
$97.50	$74.31	$193.40	$473.21
0.925	1.085	1.12	1.0875

(i)	(j)
$631.40	$1764.23
0.9125	0.125

4. Division (answers to the nearest cent)

(a) $\frac{\$57.38}{0.94}$ (b) $\frac{\$28.42}{0.88}$ (c) $\frac{\$163.20}{0.90}$ (d) $\frac{\$47.15}{0.075}$

(e) $\frac{\$6.29}{0.1025}$

(answers to the nearest $\frac{1}{10}\%$)

(f) $\frac{\$73.20}{\$850.00}$ (g) $\frac{\$25.63}{\$195.00}$ (h) $\frac{\$84.50}{\$985.00}$ (i) $\frac{\$7.24}{\$538.40}$

(j) $\frac{\$9.15}{\$239.42}$

5. Per cent (answers to the nearest cent)

(a) 6% of \$48.30 (b) $9\frac{1}{2}\%$ of \$34.20 (c) $8\frac{3}{4}\%$ of \$93.40

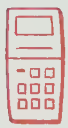
(d) 12% of \$83.52 (e) 18% of \$475.80 (f) $12\frac{1}{2}\%$ of \$593.28

(g) $16\frac{3}{4}\%$ of \$384.70 (h) 24.5% of \$259.74 (i) 107% of \$582.30

(j) 118% of \$375.25 (k) $112\frac{1}{2}\%$ of \$168.43 (l) $108\frac{3}{4}\%$ of \$437.95

$$A = P(1 + i)^n \qquad P = A \times \frac{1}{(1 + i)^n}$$

Perform the following calculations.



1. $9200(1.0935)^{10}$
2. $1600(1.1275)^{12}$
3. $44\,900(1.1125)^7$
4. $55\,700(1.135)^6$
5. $66\,900(1.0635)^8$
6. $7300 \times \dfrac{1}{(1.125)^4}$
7. $1575 \times \dfrac{1}{(1.0675)^6}$
8. $7760 \times \dfrac{1}{(1.095)^8}$
9. $8150 \times \dfrac{1}{(1.1275)^6}$

CHAPTER 14

Buying and Selling, Profit and Loss

14.1 PROFIT AND LOSS

The financial basis of retail trade, in its simplest terms, is buying goods at a cost price and then performing a service to the customer, such as display or delivery, which allows you to charge a higher selling price. This results in a gross profit on the transaction.

Two messengers cover the same route. One walks half the *time* and bicycles half the *time*. The other walks half the *distance* and bicycles half the *distance*. Who covered the route in the least time?

EXAMPLE 1. *Joseph Armas is a street vendor. Early in the morning he loads his open truck with seasonal fruit purchased at the wholesale market. Then he drives along the residential streets selling to the householders. One Saturday he purchases 200 baskets of apples at 50¢ a basket and offers them for sale at 80¢ a basket. He is charging 30¢ a basket for the service of delivering them to the purchaser's door. During the morning and early afternoon he is able to sell 165 baskets. What is his gross profit for this transaction?*

Solution

$$\begin{aligned}\text{Sales} &= 165 \times \$0.80\\ &= \$132.00\\ \text{Total cost of goods sold} &= 165 \times \$0.50\\ &= \$82.50\\ \text{Gross profit} &= \$49.50 \qquad (\$132.00 - \$82.50 = \$49.50)\end{aligned}$$

Later in the afternoon it is found that some of the apples at the bottom of the load are bruised. In order to sell the remainder quickly, the price is dropped to 45¢ a basket for the remaining 35 baskets. What is his loss on this part of the transaction?

$$\begin{aligned}\text{Sales} &= 35 \times \$0.45\\ &= \$15.75\\ \text{Total cost of goods sold} &= 35 \times \$0.50\\ &= \$17.50\\ \text{Gross profit} &= -\$1.75 \qquad (\$15.75 - \$17.50)\\ \text{Therefore loss} &= \$1.75\end{aligned}$$

What was the gross profit on the day's work?

(i) From the previous calculations:

$$\text{Gross profit} = \$49.50 - \$1.75$$
$$= \$47.75$$

(ii) As a complete question:

$$\text{Sales} = (165 \times \$0.80) + (35 \times \$0.45)$$
$$= \$132.00 + \$15.75$$
$$= \$147.75$$
$$\text{Total cost of goods sold} = 200 \times \$0.50$$
$$= \$100.00$$
$$\text{Gross profit} = \$47.75 \qquad (\$147.75 - \$100.00)$$

The above example illustrates the relation

> Sales − cost of goods sold = gross profit (if positive)
> or = loss (if negative)

EXERCISE 14-1

B **1.** Find the missing quantities

	Number of units sold	Unit cost price	Unit selling price	Cost of goods sold	Total sales	Profit (+) or loss (−)
(a)	50	$ 1.25	$1.65			
(b)	15			$82.50	$131.25	
(c)	144	0.09			17.28	
(d)	3	265.30				+$92.07
(e)	20	32.40			560.00	

2. (a) Northland Lumber sold 1000 sheets of particle board at $2.16 a sheet on the understanding that unused sheets might be returned if unmarked. After the job was completed, 45 sheets were returned. What was the net sale of particle board?
(b) Northland Lumber bought the sheets from Wabi Manufacturing at $1.98 a sheet. Find the gross profit on the sale.

3. King Book Stores Limited had gross sales of $25 000 with returns of $3750. The cost of goods sold was $13 500. Find the gross profit.

4. Walkon Shoes Limited had $8746 gross sales of shoes during their fall sale. They had returns amounting to $512 and made a gross profit of $2346. What was the cost of goods sold?

5. A shipment of 96 blouses was purchased at $4.50 each for the Christmas trade. Sales were as listed:

52 @ regular price $8.98
19 @ "Last Shopping Day" sale
  10% discount on regular price
16 @ "Boxing Day Specials"
  40% discount

The remainder at a special "Counter Soiled" sales price of $3.98.

Find the gross profit on the sales.

C **6.** (a) Find the gross profit from the following transaction: Handywhiz all purpose work bench tool was being sold at a cost price of $2.30 with a 10% discount for orders of 1000. Corner Variety ordered 1000 to take advantage of the discount. The suggested list price was $4.99, and sales were as follows:

Sales	Returns	Price	Comment
249	10	$4.49	Introductory Sale
414	29	$4.99	Suggested List Price
140		$4.49	Spring Tool Sale (no returns)
165		30% Discount on List Price	Clearance Sale
52		$2.65	Counter Soiled Sale
Remainder			Lost or Stolen

(b) What percentage gross profit was earned on the investment?

(c) What was the annual rate of return if the stock was cleared

(i) in 1 a?

(ii) in 6 mo.?

(iii) in 3 mo.?

14.2 INVENTORY AND NET PROFIT

In determining the cost of goods sold in a store, it is not sufficient to know how much the store has bought. Many of the purchased articles may still be in stock. The value of merchandise in stock is called the *inventory*.

> Cost of goods sold = (inventory at the beginning of the period) + (purchases) − (inventory at the end of the period)

Note: The cost of goods sold will include the cost of goods missing or stolen from the store, unless covered by insurance.

EXAMPLE 1. *Bits and Pieces Variety Store had an inventory on 06-01 of $25 692.50. During June they purchased goods worth $8750.00. On 07-01 the inventory was $25 963.00. What was the cost of goods sold?*

Solution

Cost of goods sold = $25 692.50 + $8750.00 − $25 963.00
= $8479.50

Sales for the month of June were $10 652.50. Find the gross profit.

Gross profit = $10 652.50 − $8479.50
= $ 2173.00

Note: This does not represent the store owner's gain for the month of June. From it must be paid store rent (or upkeep, taxes and insurance if

the store is the owner's property), services, wages and other amounts, all of which we shall call operating expenses.

$$\boxed{\text{Gross profit} - \text{Operating expenses} = \text{Net profit}}$$

A negative net profit is called a loss.

If the operating expenses of the Bits and Pieces Variety Store for June were \$1340, what was the net profit?

$$\begin{aligned}\text{Net profit} &= \$2173 - \$1340\\ &= \$833\end{aligned}$$

EXERCISE 14-2

B 1. Find the cost of goods sold in each case.

	Inventory at beginning of period	Purchases for period	Inventory at end of period
(a)	\$6275	\$30 500	\$5760
(b)	\$25 472	\$139 760	\$32 420
(c)	\$2195	\$12 090	\$3260
(d)	\$4934	\$5620	\$1498

2. Find the net profit for each of the following periods.

	Net sales	Cost of goods sold	Operating expenses
(a)	\$42 760	\$34 076	\$7620
(b)	9 425	6 792	1240
(c)	96 500	75 400	5850
(d)	27 850	22 480	2150

3. Acme Tool, during the period 07-01 to 12-31, had net sales of \$57 572. The cost of the goods sold was \$46 750, and the operating expenses were \$10 973. What was the net loss?

4. Tandem Corporation bought \$1 365 000 worth of goods and sold them for \$2 047 500. If their net profit for the year was \$417 300, find their expenses.

5. Work-Rite Tools had a 03-01 inventory of \$56 495 and a 06-01 inventory of \$47 800. Purchases for the period were \$135 870. For the same period sales were \$253 000 with operating expenses of \$57 200.
Find
(a) The cost of goods sold (b) Gross profit for the period
(c) Net profit for the period.

6. Uni-Hold Fasteners had the following sales and inventory figures for four quarters:

	19-		1 Inventory as of date	2 Purchases previous quarter	3 Sales previous quarter	4 Operating Expenses prev. quarter	5
1	01	01	12 400				
2	04	01	15 300	13 400	18 900	3 360	
3	07	01	14 100	12 900	25 300	5 240	
4	10	01	17 500	15 300	21 400	5 700	
5	01	01	13 700	9 700	24 300	6 480	

(a) Find the net profit for the year.
(b) Find the net profit for the year as a percentage of sales.

	19-		1 Inventory	2 Semi-annual purchases	3 Semi-annual sales	4 Semi-annual operating expenses	5
1	03	01	7 450				
2	09	01	6 230	25 940	46 800	7 120	
3	03	01	8 190	32 860	40 650	6 090	

7.
(a) Find the net profit for each half year.
(b) Find the year's profit as a percentage of the year's sales.

14.3 PROFIT AND LOSS STATEMENT

Margaret Hastings was considering the purchase of King Korner Stores. In order to assess the value of the business, she asked to be shown the profit and loss statement for the past fiscal year. A fiscal year is the financial year of a business at the end of which accounts are balanced.

To get a better idea of the efficiency of the operation and the return she might expect from her investment, Ms Hastings expressed the cost of goods sold, gross profit, operating expenses and net profit each as a percentage of the net sales.

Use five 1's and signs of operations to make a mathematical expression equal to 100.

King Korner Stores, Limited
Statement of Profit and Loss
(For the year ending 1971-03-31)

Net sales	$125 900
Cost of goods sold	81 835
Gross profit	44 065
Operating expenses	25 180
Net profit	$ 18 885

$$\text{Cost of goods sold} = \frac{81\,835}{125\,900} \times 100\% = 65\% \text{ of net sales}$$

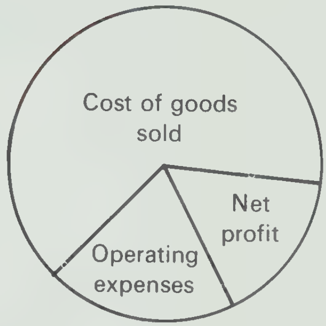
$$\text{Operating expenses} = \frac{25\,180}{125\,900} \times 100\% = 20\% \text{ of net sales}$$

The profit and loss statement now looks like this:

King Korner Stores, Limited
Statement of Profit and Loss
(For the year ending 1971-03-31)

Net sales	$125 900	100%
Cost of goods sold	81 835	65%
Gross profit	44 065	35%
Operating expenses	25 180	20%
Net profit	$ 18 885	15%

NET SALES



EXERCISE 14-3

B 1. For each of the following, calculate the gross profit and net profit and show all quantities as a percentage of net sales.

	Net sales	Cost of goods sold	Operating expenses
(a)	$ 5 964	$ 4 055.52	$ 1 073.52
(b)	70 560	50 803.20	7 056.00
(c)	924 600	536 260.00	258 888.00
(d)	8 256	5 201.28	1 816.32
(e)	15 620	10 465.40	2 811.60

2. Draw up a statement of profit and loss for Uptown Stores Limited for the year ending 1974-12-31. Show all amounts as a percentage of net sales. Net sales were $86 000, cost of goods sold was $84 160 and operating expenses were $24 080.

3. Manufacturers' Agents Incorporated buy surplus or "end of line" articles and sell them in a large building where they hold "warehouse sales", keeping overhead to a minimum. Draw up a statement of profit and loss if sales were $179 000, cost of goods sold was $100 240 and operating expenses were $8055.

4. Draw up a statement of profit and loss for Q.P. Sales and Service for the period 06-01–09-01.
Inventory at start of period $15 240

Inventory at end of period	12 960
Cost of goods purchased	20 300
Net sales	43 700
Operating expenses	17 480

5. Draw up a profit and loss statement for the period 07-01–01-01 for Sports Equipment Limited.

Inventory at start of period	$ 7 890
Inventory at end of period	8 430
Cost of goods purchased	14 210
Net sales	21 872
Operating expenses	5 450

6. Draw up a profit and loss statement for the period 01-01–07-01 for Sports Equipment Limited.

Inventory at start of period	$ 8 430
Inventory at end of period	9 876
Cost of goods purchased	16 850
Net sales	24 650
Operating expenses	6 280

7. You are considering the purchase of a small variety store. Two possible businesses supply you with the following data for the period 03-01–06-01.

	Business A	Business B
Inventory at start of period	$47 600	$12 150
Inventory at end of period	39 450	9 500
Cost of goods purchased	5 200	15 000
Net sales	26 700	28 240
Operating expenses	10 100	6 200

Draw up a statement of profit and loss for each. If the prices are comparable, which business seems to be the wiser investment?

14.4 MARGIN AND MARKUP

Gross profit is often referred to by two different names, gross margin and markup. These two names for the same quantity arise from two needs of the merchant.

From an estimate of future sales and knowledge of what his expenses will be, the merchant can calculate what margin in the selling price is required to meet these expenses and give an acceptable level of profit. The margin is usually expressed as a percentage of the selling price although the term is also used to refer to the amount of money involved. Where ambiguity occurs we shall use the term "rate of margin" to refer to the percentage.

The number 47974 is palindromic, that is, it reads the same both ways. What is the next greatest palindromic number?

$$\text{Gross margin} = \text{selling price} - \text{cost price}$$

$$\text{Rate of margin} = \frac{\text{selling price} - \text{cost price}}{\text{selling price}} \times 100\%$$

When goods are bought, the merchant must be able to quickly set the price at which they are to be sold. Since the cost price is known, the fastest method of setting the selling price is to know the amount by which the cost price is to be raised or "marked up".

$$\text{Selling price} = \text{cost price} + \text{markup}$$
$$\text{Markup} = \text{selling price} - \text{cost price}$$
$$\text{Rate of markup} = \frac{\text{selling price} - \text{cost price}}{\text{cost price}} \times 100\%$$

EXAMPLE 1. *Highway Tire Shop can sell a top grade snow tire for \$27.95 and meet the competitive price. It has been found that a margin of 35% is required to run the business and give a reasonable profit. How much could Highway Tire pay a wholesaler for the tires?*

Solution

(i) Gross Margin $= 0.35 \times \$27.95$ or
$= \$9.78$
Cost price $= \$27.95 - \$9.78$
$= \$18.17$

(ii) Gross Margin $= 35\%$
Cost $= 100\% - 35\%$
$= 65\%$
Cost price $= 0.65 \times \$27.95$
$= \$18.17$

EXAMPLE 2. *If the rate of markup on an article is 45%, and the cost price is \$4.75, find the selling price.*

Solution

$$\begin{aligned}\text{Selling price} &= \text{cost price} + \text{markup}\\ &= 145\% \text{ of cost price}\\ &= 1.45 \times \$4.75\\ &= \$6.89\end{aligned}$$

EXAMPLE 3. *Northern Ski Shop must make a margin of 40% on sales to make a profit. What markup should be put on skis with a cost price of \$45?*

Solution Let the markup in dollars be x.

$$\begin{aligned}\text{Selling price} &= \text{cost price} + x\\ &= 45 + x\end{aligned}$$

Since the margin is 40% of the selling price, margin $= 0.40(45 + x)$.
Since markup and margin refer to the same quantity, gross profit, then

$$\begin{aligned}x &= 0.40(45 + x).\\ x &= 18 + 0.4x\\ 0.6x &= 18\\ x &= 30\end{aligned}$$

Therefore the markup is \$30.

Note: For a given article with a fixed price and cost expressed as an amount of money, margin = markup; as a percentage, rate of margin $\neq$ rate of markup.

EXERCISE 14-4

A 1. Complete the given table.

	Cost price	Markup	Selling price
(a)	\$22.43	\$1.15	
(b)	\$26.80	\$5.90	
(c)	\$8.30		\$10.50
(d)	\$164.00		\$205.00
(e)		\$32.60	\$174.60
(f)		\$3.40	\$24.60

2. Complete the given table.

	Selling price	Margin	Cost price
(a)	\$48.50	\$12.25	
(b)	\$65.30	\$15.20	
(c)	\$152.10		\$121.00
(d)	\$94.20		\$42.15
(e)		\$9.80	\$17.20
(f)		\$17.30	\$38.40

3. Find the selling price.

	Cost price	Rate of markup	Selling price
(a)	\$4.00	10%	
(b)	\$25.00	20%	
(c)	\$15.00	30%	
(d)	\$100.00	75%	
(e)	\$80.00	25%	
(f)	\$120.00	40%	

4. Find the cost price.

	Selling price	Rate of margin	Cost price
(a)	\$7.00	20%	
(b)	\$12.00	25%	
(c)	\$50.00	40%	
(d)	\$140.00	50%	
(e)	\$60.00	30%	
(f)	\$90.00	10%	

B **5.** Find the missing quantity.

	Cost price	Markup	Rate of markup	Selling price
(a)	$36.40	$16.38		
(b)	$9.20		40%	
(c)		$2.50		$6.00
(d)	$1793.00			$2546.06

6. Find the missing quantities.

	Cost price	Margin	Rate of margin	Selling price
(a)		$8.81		$27.53
(b)			30%	$5.70
(c)		$13.65	42%	
(d)	$4328.90			$5930.00

7. A.B.C. Stores require a margin of $33\frac{1}{3}\%$ to operate at a profit. They can sell a line of toy trucks for $4.65 each. How much can they afford to pay for them? Calculate the rate of mark-up.

8. Lo-Cost Store has annual operating expenses budgeted at $75 000. The owner expects a net profit of 12% on estimated annual sales of $280 000.

(a) If the sales and expenses estimated are correct what will the cost of goods sold be?

(b) What rate of markup is used?

(c) What rate of margin is used?

C **9.** Smith's Hardware sells hockey sticks which cost the store $1.30 each. If the margin is 35%, at what price should the sticks be sold?

10. If outboard motors bought for $486 are sold at a margin of 28%, find the selling price.

14.5 CONVERTING MARGIN AND RATE OF MARKUP

EXAMPLE 1. *If the rate of margin is 35% calculate the rate of markup.*

Solution Since we are dealing with rates the answer will be independent of the actual selling price.
Let the selling price be 1 unit.

$$\text{Margin} = 35\% \text{ of selling price}$$
$$= 0.35$$
$$\text{Cost price} = \text{selling price} - \text{margin}$$
$$= 1.00 - 0.35 \qquad (100\% - 35\%)$$
$$= 0.65$$
$$\text{Markup} = \text{margin}$$

From this example we see that rate of mark-up
$= \dfrac{\text{rate of margin}}{100\% - \text{rate of margin}} \times 100\%$

$$= 0.35$$

$$\text{Rate of markup} = \frac{\text{markup}}{\text{cost price}} \times 100\%$$

$$= \frac{0.35}{0.65} \times 100\%$$

$$= 53.8\%$$

EXAMPLE 2. *If the rate of markup is 40%, find the rate of margin.*
Solution Let the cost price be 1 unit.

$$\text{Markup} = 40\% \text{ cost price}$$

$$= 0.40$$

$$\text{Selling price} = \text{cost price} + \text{markup}$$

$$= 1.40$$

$$\text{Margin} = \text{markup}$$

$$= 0.40$$

$$\text{Rate of margin} = \frac{\text{margin}}{\text{selling price}} \times 100\%$$

$$= \frac{0.40}{1.40} \times 100\%$$

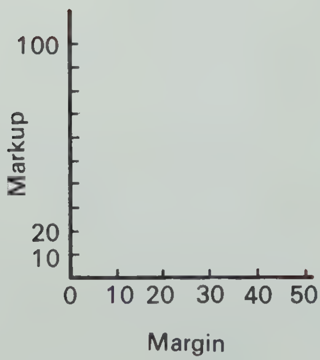
$$= 28.6\%$$

From this example we see that rate of margin $= \frac{\text{rate of markup}}{100\% + \text{rate of markup}} \times 100\%$

EXERCISE 14-5

B

1. To quickly convert margin and markup, a graph would be useful.
Complete the following table and plot the values with *margin* on the horizontal axis and *markup* on the vertical axis. Calculate all values to one-tenth of 1%.



Margin (%)	Markup (%)
	0
	10
	20
	30
28.6	40
	50
35	53.8
40	
45	
48	
50	

2. Use your graph to make the following conversions.
(a) Margin of 24% to markup. (b) Margin of 32% to markup.
(c) Markup of 48% as margin. (d) Markup of 65% as margin.

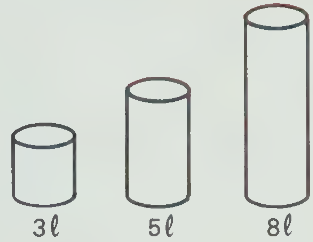
3. A hardware store works on a margin of 37%. What is the markup on a chain saw which wholesales at $165?

4. A toy store buys trucks at $3.50 and sells them with a markup of 46%. Find the margin and the selling price.

5. Watches bought for $7 are sold at $12.50 regular with a 12% discount during the Christmas Sale. What are the sale price and the rate of margin?

6. A store owner estimates that the year's sales will be $150 000. From his estimates of costs and expenses, he calculates that he must work on a margin of 27%.
(a) What is his rate of markup?
(b) What is the selling price of an article costing $76.80?
(c) Find his gross profit for the year if his estimate is correct.

If the 8 ℓ jug is full of water, use the empty 3 and 5 ℓ containers to divide the water into two 4 ℓ amounts.



C **7.** A store owner estimates his operating expenses to be $46 500 for the coming year. In addition he wishes to pay himself $15 000 salary for his work plus 8% on his investment of $130 000 in stock and equipment:
(a) What gross profit must the business produce?
(b) If the store can operate competitively on a margin of 18%, what sales must the business have to fulfill the owner's expectations?

8. A store owner invests $65 000 in stock. The store works on a margin of 21% with operating expenses of $21 000/a.
(a) If his stock *turns over* in 4 mo., find his annual net profit.
(b) If he reduced his turnover time to 3 mo., what would the net profit be?

Four months sales equals the total inventory.

14.6 TRADE DISCOUNTS

When a contractor building houses buys material from a lumber yard, he will not usually pay the list price, which is the full price of the goods. The reduction in price which the contractor receives is called a trade discount, and the price he pays for the articles is the net price.

This gives us the relation

$$\boxed{\text{Net price} = \text{list price} - \text{discount}}$$

The rate of trade discount is expressed as a percentage of the list price.

EXAMPLE 1. *Find the net price of goods with a list price of $475.60 and a trade discount of 30%.*

Solution

(i) Trade discount = $475.60 × 0.30
= $142.68
Net price = $475.60 − $142.68
= $332.92

(ii) Trade discount = 30% of list price
Net price = (100% − 30%) of list price

$$= 70\% \text{ of list price}$$
$$= 0.70 \times \$475.60$$
$$= \$332.92$$

For a variety of reasons, such as meeting competition or giving special consideration to customers who buy large quantities of materials, a second trade discount may be applied to the net price after the first discount has been taken.

EXAMPLE 2. *Find the net price of goods with a list price of $927 after applying trade discounts of 25% and 15%.*

Solution

(i)

$$\begin{aligned}
\text{1st discount} &= \$927.00 \times 0.25 \\
&= \$231.75 \\
\text{1st discounted price} &= \$927.00 - \$231.75 \\
&= \$695.25 \\
\text{2nd discount} &= \$695.25 \times 0.15 \\
&= \$104.29 \\
\text{Net price} &= \$695.25 - \$104.29 \\
&= \$590.96
\end{aligned}$$

or (ii)

$$\begin{aligned}
\text{1st discount} &= 25\% \text{ of list price} \\
\text{1st discounted price} &= (100\% - 25\%) \text{ of list price} \\
&= 75\% \text{ of list price} \\
\text{2nd discount} &= 15\% \text{ of 1st discount price} \\
\text{Net price} &= (100\% - 15\%) \text{ of 1st discount price} \\
&= 85\% \times 75\% \text{ of list price} \\
&= 0.85 \times 0.75 \times \$927.00 \\
&= \$590.96
\end{aligned}$$

EXERCISE 14-6

B **1.** Find the net price after applying the following discounts.

	List price	1st discount	2nd discount
(a)	$ 75.40	8%	nil
(b)	189.20	6.5%	nil
(c)	527.80	8%	3%
(d)	1857.00	10%	2.5%
(e)	5163.00	12%	1.5%

2. Alpha Printing gives Beta Stationery Store trade discounts of 8% and 3% on all printing orders handled in the shop. Find the net price on an order which lists at $375.

3. Acme Plumbers can buy pipe from Copper Tube Company at 40¢/m with discounts of 15% and 2%, or from Alloy Pipe Company at 35¢/m

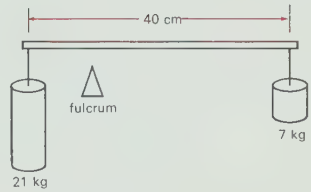
with discounts of 10% and $1\frac{1}{2}$%. Compare the list and net prices on orders for 12 000 m.

4. In each of the following, determine which gives the lower net price by calculating each amount.
(a) $12.65 list with discounts of 22% and 5% or $13.05 list with discounts of 25% and 4%.
(b) $475 list with discounts of 30% and 12% or $496 list with discounts of 35% and 8%.

5. A radio which lists at $34.50 is sold with discounts of 25% and 10%.
(a) Find the net price.
(b) Calculate the net price when a single discount of 35% is given.
(c) Are the two equivalent? If not, which gives a lower net price, successive discounts or a total rate of discount?

6. A saw lists for $14.75 with a trade discount of 20% to a carpenter.
(a) What is the net price of the saw?
(b) If a sale discount of 11% is also offered, find the sale price.

7. (a) Hot Shot Hardware offers a trade discount of 15% with a second sale discount of 10%. Find the net price of an article listing for $76.40.
(b) Homeway Hardware offers a trade discount of 10% with a second sale discount of 15%. Find the net price of an article listing for $76.40. Does the order of discounts affect the net price?



Place the fulcrum so the bar balances.

14.7 EQUIVALENT SINGLE DISCOUNT

To compare various competitive offers, we express a series of trade discounts as a single equivalent discount.

EXAMPLE 1. *Find the single discount equivalent to successive discounts of 20% and 15%.*

Solution

1st trade discount = 20% of list price
1st discounted price = 80% of list price
2nd discount = 15% (1st discounted price)
$\therefore$ Net price = 85% (1st discounted price)
= 85% (80% of list price)
= 0.85 × 0.80 × 100% of list price
= 68% of list price
$\therefore$ Equivalent single discount = 100% − 68% = 32%

Note: This is not equal to the sum of the discounts.

EXAMPLE 2. *Find the single discount equivalent to successive discounts of 25%, 15%, and 10%.*

Solution

100% − 25% = 75%

$$100\% - 15\% = 85\%$$
$$100\% - 10\% = 90\%$$
$$\therefore \text{Net price} = 0.75 \times 0.85 \times 0.90 \times 100\% \text{ of list price}$$
$$= 57.375\% \text{ of list price}$$
$$\therefore \text{Equivalent single discount} = 100\% - 57.375\%$$
$$= 42.625\%$$

EXERCISE 14-7

B **1.** Find the single discount which is equivalent to the following successive discounts.
(a) 20% and 15%
(b) 25% and 16%
(c) 17% and 8%
(d) 25%, 16%, and 9%
(e) 35%, 20%, and 5%
(f) 22%, 12%, and 6%

2. It is advantageous to work out a single discount which is equivalent to a series of discounts when the discounts are to be applied to a number of articles. Find the net price corresponding to the following list prices if they are subject to discounts of 30%, 14%, and 2%.
(a) \$729.30
(b) \$82.95
(c) \$17.24
(d) \$1097
(e) \$4.95
(f) \$27.40

3. Playfair Company allows trade discounts of 28% and 12% while Axiom Incorporated gives discounts of 25%, 14%, and 2%. If the list prices for similar products are the same, compare net prices by showing each series of discounts as a single equivalent discount.

4. Compare the effective discounts offered under the following plans.
(a) 12%, 8%, and 4%; or 15%, 5%, and 5%
(b) 30%, 10%, and 2%; or 25%, 10%, and 6%

5. You are preparing an order for a number of items to come from two suppliers.

Supplier A gives you discounts of 20% and 4%.

Supplier B gives you discounts of 15% and 8%.

Compare the net prices for the following items; prepare an order list for each company and give the total amount of each order.

Item #	Company A List price	Company B List price
1	\$47.20	\$45.30
2	\$172.00	\$160.00
3	\$14.50	\$15.10
4	\$493.00	\$487.00
5	\$8.43	\$8.25
6	\$15.30	\$17.20
7	\$4.85	\$4.85
8	\$17.80	\$16.40
9	\$85.90	\$86.20
10	\$581.00	\$576.00

14.8 DETERMINING THE RATE OF TRADE DISCOUNT

EXAMPLE 1. *Tip Top Tap Company lists standard sink faucets at \$4.60 each. Their chief competitor, Fine Faucets Inc., sells to the trade at \$2.76 for the same grade tap. What rate of trade discount should Tip Top offer to meet the price of competition?*

Solution

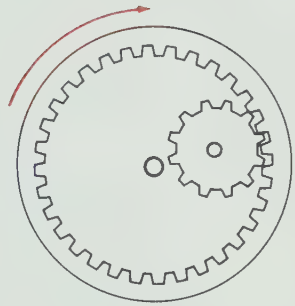
$$\text{List price} = \$4.60$$
$$\text{Net price} = \$2.76$$
$$\text{Discount} = \$4.60 - \$2.76 = \$1.84$$
$$\text{Rate of trade discount} = \frac{1.84}{4.60} \times 100\%$$
$$= 40\%$$

EXAMPLE 2. *A manufacturer of paint sells flat wall paint at a list price of \$5.20 for 5 ℓ with a trade discount of 25%. To meet competition, the price must be reduced to \$3.51. What additional successive discount should be given?*

Solution

$$\text{List price} = \$5.20$$
$$\text{1st discounted price} = 75\% \times \$5.20$$
$$= \$3.90$$
$$\text{Net price} = \$3.51$$
$$\text{2nd discount} = \$0.39$$
$$\text{Rate of 2nd discount} = \frac{0.39}{3.90} \times 100\%$$
$$= 10\%$$

Note: Since we are calculating successive discounts, the second discount is found as a percentage of the first discount price.



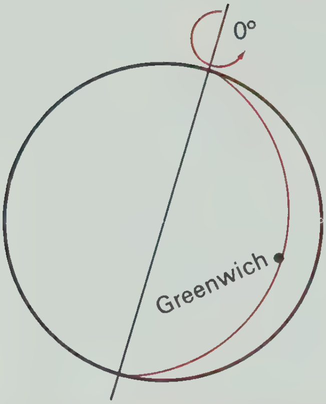
The large wheel rotates in the direction shown and has 32 teeth. The small wheel has 12 teeth. If the large wheel rotates at 300 r/min, give the direction and speed of rotation of the small wheel.

EXERCISE 14-8

B **1.** Find the single discount rate required to reduce each given list price to the net price shown.

	List price	Net price
(a)	\$ 576.50	\$ 449.67
(b)	1073.00	912.05
(c)	5872.00	4815.04
(d)	3.75	3.15
(e)	94.20	89.96

2. Standard Brick sells tiles at 12¢ each with a trade discount of 8% on orders over 1000 tiles. What successive discount rate must be offered to meet a competitive price of 10.5¢ per tile?



How many degrees does the earth rotate through in 1 h? If the sun is directly overhead when the time is 16:00 in Greenwich what is your longitude?

3. What successive discount rates should be given after a discount of 15% in order to reduce a price of \$36.80 to \$29.72?

4. What discount rate must follow a discount of 26% if the successive discounts are to be equivalent to discounts of 30% and 15%?

5. A shower stall unit sells for \$53.40 with a trade discount of 20%.
(a) What is the net price of the unit?
(b) What second discount rate must be offered to meet a competitor's sale price of \$38.30?

6. A fireplace unit has a list price of \$265.00 with a trade discount of 25%. What second discount rate is required to reduce the price to \$185.00?

7. A carpet store has three classes of customers: casual customers, contractors spending less than \$10 000/a, and contractors spending more than \$10 000/a. The first class of customer pays list price, the second receives a trade discount, and the third receives the trade discount and a second "quantity purchase" discount.
(a) The three prices on a particular rug are \$6.20/m², \$4.15/m², and \$3.90/m². Find the rates of discount.
(b) If the prices for **7.** (a) had been
(i) \$8.50, \$6.37$\frac{1}{2}$, and \$5.61
(ii) \$10.40, \$7.28, and \$6.84
(iii) \$12.10, \$9.56, and \$8.79
(iv) \$27.30, \$19.66, and \$17.49
find the rates of discount to the nearest percent in each case.

14.9 CASH DISCOUNT

To encourage prompt payment of accounts, many companies give a discount for payment within a specified period. If payment is delayed beyond a further allowed time, the company may charge interest on the overdue account. A commonly used discount plan is 2/10, n/30 which indicates that a discount of 2% is given on accounts paid within 10 d, and the net price must be paid within 30 d or interest will be charged.

EXAMPLE 1. *Corner Groceries purchased 5 cases of dills at \$4.50 each and 7 cases of hot dog relish at \$8.40 each on 1976-06-19. Find the net cash price if the invoice was paid 1976-06-26 with terms of 2/10, n/30.*

Solution

$$\begin{aligned} 5 \times \$4.50 &= \$22.50 \\ 7 \times \$8.40 &= \underline{\$58.80} \\ \text{Net price} &= \$81.30 \end{aligned}$$

Since payment has been made within 10 d,

$$\begin{aligned} \text{Rate of cash discount} &= 2\% \\ \text{Net cash price} &= 98\% \times \$81.30 \\ &= \$79.67 \end{aligned}$$

Often a cash discount is offered in conjunction with other discounts.

#62573

Peter Piper Pickle Co.
Farmtown, Alta.

Date: 1976-06-19
Terms: 2/10, n/30

To: Corner Groceries
Whitefield, Alta.

Cases	Description	Price	Amount
5	Dills 426 ml 24's	$4.50	$22.50
7	Hot dog relish 341 ml 36's	8.40	58.80

EXAMPLE 2. *Builders' Supply Company sells Home Construction Company 200 bags of cement at $1.85 each with trade discounts of 15% and 5%, terms 2/10, n/30. Find the net cash price if the invoice was paid within 10* d.

Solution

Gross amount = $1.85 × 200 = $370

Net cash price = $370.00 × 0.85 × 0.95 × 0.98

= $292.80

$$\left\{\begin{array}{l} 100\% - 15\% = 85\% \\ 100\% - 5\% = 95\% \\ 100\% - 2\% = 98\% \end{array}\right.$$

EXERCISE 14-9

B **1.** Find the net cash price for the following amounts if paid within the discount period.

	Gross amount	Trade discount	Terms
(a)	$ 523.60	15%	2/10, n/30
(b)	7846.00	8%	3/10, n/30
(c)	5800.00	12%, 4%	$1\frac{1}{2}$/10, n/30
(d)	24.60	20%	2/10, n/30
(e)	129.30	10%, 5%	3/10, n/30

2. Omega Electronics Limited sells to Corner Appliances Limited three television sets at $250.00, six radio-phonograph combinations at $375.00 and 20 transistor radios at $14.50. Apply a trade discount of 20% and a cash discount of 2% to the total sale. What was the cash discount price?

3. Find the cash price for an order listing at $5280.00 with a trade discount of 20% and a cash discount of 2%.

4. John's Plumbing sells to the trade at 23% trade discount with terms 2/10, n/30, 18%/a interest charged on overdue accounts. An order to a tradesman for $674.00 list is charged on 06-26. Find the amount due.
(a) 06-30 (b) 07-16 (c) 08-30

The Canada–U.S. border across the prairie is at 49°N latitude. If the radius of the earth is 6400 km how far is the border from the equator?

5. A company has annual purchases of $250 000.
(a) If the average discount offered for payment within 10 d is $1\frac{3}{4}$%, how much can they save in a year?
(b) If money is worth 12%/a how much can they earn by consistently not paying until the 10th day?

C 6. A company buys $56 000 worth of goods with terms 2/10, n/30. The company will not have funds available to pay for the goods until the thirtieth day. How much money would they save if the money to pay the account was borrowed from the bank at 12% on the tenth day (in order to take advantage of the discount) and was repaid to the bank on the thirtieth day?

14.10 SALES TAX

1973 Table of Provincial Sales Taxes

New Brunswick	8%
Quebec	8%
Newfoundland	7%
Nova Scotia	7%
Ontario	7%
Prince Edward Island	7%
British Columbia	5%
Manitoba	5%
Saskatchewan	5%
Alberta	0%

When certain goods are manufactured in Canada, the manufacturing company is licensed by the Federal Government to collect a 12% sales tax and pay this to the government monthly. Some classes of goods, such as farm machinery, are exempt. Since this tax is collected by the manufacturer, it is included in the price of the article. Federal sales tax is paid on the wholesale price after any applicable discounts have been taken.

Many provinces also have a sales tax. Since the provincial sales tax is collected by the retailer, it is not included in the marked price but is paid in addition to it.

EXAMPLE 1. *Auto Game Company sells 50 auto race sets at $35 with a trade discount of 15% and a cash discount of 2%. Federal sales tax is 12%. Find the invoiced amount.*

Solution

$$\begin{aligned}\text{Gross price} &= 50 \times \$35\\ &= \$1750\\ \text{Net price} &= \$1750 \times 85\% \times 98\%\\ &= \$1457.75\\ \text{Sales tax} &= \$1457.75 \times 12\%\\ &= \$174.93\\ \text{Invoiced amount} &= \$1457.75 + \$174.93\\ &= \$1632.68\end{aligned}$$

EXERCISE 14-10

B 1. Find the price to the retailer (including sales tax) for the following sales direct from the manufacturer. Federal sales tax is 12%.
(a) $478.54 (b) $7372.50 (c) $5016
(d) $895 (e) $7492

Provincial tax is calculated on the price with the federal tax added.

2. Find the retail price plus tax of the following goods sold directly from the manufacturer to the customer. Federal sales tax is 12% and the provincial sales tax is 7%.

(a) $72.50 (b) $486 (c) $183
(d) $2.40 (e) $16.50

3. Playtime Manufacturing Company Limited produces Buildblocks for $2.50 each, which includes their profit. The blocks are sold to Toy Distributing Limited for $2.50 plus 12% sales tax. Toy Distributing sells them to retail outlets after adding a markup of 15%. To meet the "suggested retail price", the retailer marks his cost up 50%. When you buy a Buildblock, you pay the retail price plus 7% provincial sales tax. What is your cost?

4. When discounts are given, the sales tax is charged on the net price. Find the final price, including tax, of the following.

	List price	Discounts	Sales tax
(a)	$76.50	20%	12%
(b)	$135.20	15%, 2%	12%
(c)	$5163.00	40%, 2%	7%
(d)	$39.20	35%, 15%	7%
(e)	$394.00	25%, $1\frac{1}{2}$%	7%
(f)	$87.50	12%, 2%	12%, 8%
(g)	$245.00	18%, 7%	11%, 7%
(h)	$24.50	20%, 3%	12%, 5%

REVIEW EXERCISE

1. Find the net profit on each of the following transactions.
(a) Bought: 750 articles at $32.60 each, total sales $36 000, with operating expenses of $9000.
(b) Bought: 500 articles for a total cost of $25 600, sold at $69.20, operating expenses are $1200.

2. (a) Ace Variety Store, for the period of 01-01–06-30, had an opening inventory of $12 750. Goods were bought at a value of $56 000, and the closing inventory of the store was $14 850. Sales for the period were $80 000, with operating expenses of $17 000. What was the net profit?
(b) Draw up a profit and loss statement for the data in part (a). Show all quantities as a percentage of net sales.

3. Find the cost price of an article sold for $76.40, with a margin of 25%.

4. What is the selling price of an article with a cost price of $3.65 and a rate of markup of 40%?

5. What is the markup on an article with cost price of $23.50 and a margin of 22%?

6. (a) If a company is working on a margin of 24%, what is their rate of markup?
(b) If the rate of markup is 42%, what is the margin?

7. Find the rate of markup and rate of margin on the retail price and the rate of discount for each of the following transactions.

	Cost to retailer	Retail price	Discount price
(a)	$40.00	$50.00	$45.00
(b)	$104.00	$130.00	$117.00
(c)	$4.71	$7.85	$7.22
(d)	$35.50	$62.13	$52.81
(e)	$2560.00	$3840.00	$3264.00

8. Give the net price of goods with a list price of $72.46 and trade discounts of 20% and 12%.

9. What is the single discount equivalent to successive discounts of 28% and 15%?

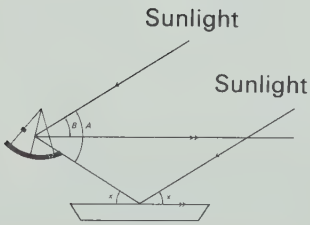
10. A tent lists for $42.50 in the sporting goods catalogue. The company offers a club discount of 20% to members of a camping club. What second discount must be given to reduce the price to $27.88?

11. An article is offered for sale with a list price of $34.20, trade discounts of 15% and 8% and terms 2/10, n/30. What is the net cash price?

12. Alumstorm Company Limited makes storm windows and storm doors for home installation. A 3 cm storm door retails for $24.50 plus 12% federal sales tax and 5% provincial sales tax. A contractor is given a trade discount of 15% and terms of 2/10, n/30 applied before the calculation of the tax. What is the net cash price plus tax?

13. A stereo lists at $245.00 with a trade discount of 18% and sales tax of 7%. Find the cost to the purchaser.

14. A manufacturer fabricates toy cars at a cost of $0.73 each. The following sequence of factors determines the price:
Manufacturers markup 40%, federal sales tax 12%, retailer's markup 50%, sale discount 20%, provincial sales tax 8%.
Find the cost to the consumer.



At sea, the distant horizon gives a horizontal line above which the angle of the sun can be measured ($\angle B$). On land, hills and trees make this impossible. A pan of mercury gives a horizontal reflecting surface. Show that $\angle A$ between the sun and its reflection in the pan is twice $\angle B$

REVIEW AND PREVIEW TO CHAPTER 15

EXERCISE 1 *Basic Operations*

1. Add

(a)	(b)	(c)	(d)
3.256	325.600	0.257	285.700
25.270	61.250	21.635	28.350
0.635	45.267	6.250	0.275
21.600	312.700	512.700	30.010
	8.400	28.732	624.800

2. Subtract

(a)	(b)	(c)	(d)
358.25	6.275	58.75	0.03741
211.73	4.712	32.97	0.01248

3. Multiply

(a)	(b)	(c)	(d)
358.2	0.2175	26.85	0.625
$\times$0.75	$\times$3.14	$\times$5.75	$\times$0.037

4. Divide to three significant figures.

(a) 358.7 by 2.75 (b) 0.2753 by 4.612
(c) 38.25 by 0.281 (d) 3.215 by 81.6

EXERCISE 2 *Order*

1. Express the following sets of numbers from smallest to largest.

(a) 15, 22, 17, 46, 21, 9, 85, 11, 28, 20, 75
(b) 33, 24, 56, 33, 29, 85, 24, 47, 31, 83, 85, 9
(c) $\frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{7}{16}, \frac{1}{4}, \frac{3}{16}, \frac{5}{32}$

EXERCISE 3 *Averages*

1. Find the arithmetic mean average of the following sets of numbers.

(a) 85, 71, 39, 49, 68, 53, 72, 65.
(b) 58, 78, 47, 58, 63, 54, 79, 61.
(c) 68.4, 21.3, 58.5, 63.2, 71.7, 58.5, 48.6.
(d) 328, 527, 683, 425, 501, 575.

Evaluate.

1. $\sqrt{385.6}$
2. $\sqrt[4]{27.69}$
3. $\sqrt[8]{583.2}$
4. $\sqrt[4]{0.8176}$
5. $\sqrt[8]{6.8173}$
6. $(\sqrt{81.76})^3$
7. $(\sqrt[4]{156.8})^3$
8. $\sqrt{(94.32)^2}$
9. $\sqrt{(0.5813)^3}$
10. $\sqrt[4]{(56.48)^3}$

CHAPTER 15

Statistics

15.1 MEASURES OF CENTRAL TENDENCY

Mean (Arithmetic)

The **mean** of a set of values is the sum of the values divided by the number of values in the set. It is usually represented by $\bar{x}$ (x-bar).

For a set of n values $x_1, x_2, x_3, \ldots x_n$

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \ldots x_n}{n}$$

Median

The **median** (M) of a set of ordered values is the middle value.

For a set of n ordered values $x_1, x_2, x_3, \ldots x_n$

1. If n is odd $\quad M = x_{\frac{(n+1)}{2}}$
2. If n is even $\quad M = \dfrac{x_{n/2} + x_{(n+2)/2}}{2}$

Mode

The mode of a set of values is the value which occurs most frequently.

Measure of Dispersion

Range

The range (R) for a set of values is the difference between the largest and the smallest value.

For a set of n ordered values $x_1, x_2, x_3, \ldots x_n$

$$R = x_n - x_1$$

EXAMPLE 1. *Find the mean, median, mode, and range for the set of values 14, 25, 17, 15, 21, 20, 18, 20, 16, 23.*

Solution
Mean: There are 10 values

$\therefore n = 10$

$$\therefore \bar{x} = \frac{14 + 25 + 17 + 15 + 21 + 20 + 18 + 20 + 16 + 23}{10}$$

$$= \frac{189}{10}$$

$$= 18.9$$

The mean is 18.9.

Median: To find the median the values must be ordered, usually from smallest to largest as in Table 15-1.

$$n = 10$$

$$\therefore \frac{n}{2} = 5 \quad x_5 = 18$$

$$\frac{n+2}{2} = 6 \quad x_6 = 20$$

$$M = \frac{18 + 20}{2}$$

$$= 19$$

The median is 19.

x_1	14
x_2	15
x_3	16
x_4	17
x_5	18
x_6	20
x_7	20
x_8	21
x_9	23
x_{10}	25

Table 15-1

Mode: Since 20 occurs more frequently than any other value (twice), the mode is 20.

Range: $x_{10} = 25 \quad x_1 = 14$

$$R = 25 - 14$$

$$= 11$$

The range is 11.

EXERCISE 15-1

B Find the mean, median, and range for the following sets of data. Find the mode where one exists.

1. 12, 17, 24, 13, 16

2. 3, 9, 8, 4, 7, 6, 4

3. 5, 7, 7, 8, 8, 8, 9, 9, 10

4. 42, 58, 64, 65, 53, 71, 66, 47

5. 135, 256, 191, 241, 176, 204

6. 58.3, 57.6, 58.2, 58.0, 57.9

7. 0.040, 0.070, 0.095, 0.080, 0.055
0.045, 0.070, 0.090, 0.075, 0.065

8. 5.2, 3.1, 7.6, 5.9, 4.7, 4.8
6.5, 5.7, 3.4, 7.7, 5.9, 6.3

9. 24.6, 95.4, 14.8, 38.9, 97.3

The hands of a clock directly overlap at 12:00. In how many minutes will they be in a straight line?

11.7, 43.3, 93.8, 49.5, 36.7

10. 0.00, 0.17, 0.34, 0.50, 0.64
0.77, 0.87, 0.93, 0.98, 1.00

11. A class of 20 students has the following marks out of 25 questions on a test.

22, 25, 20, 19, 12, 16, 20, 17, 23, 19,
18, 16, 13, 10, 15, 17, 21, 18, 16, 23

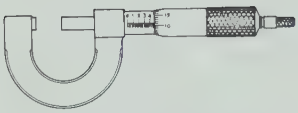
(a) Find the median mark, mean, range.

A second class of 19 students has the following marks on a similar test.

20, 19, 17, 18, 13, 14, 16, 18, 22, 17,
18, 16, 24, 14, 16, 21, 15, 19, 15

(b) Find the median mark, mean, range.

(c) If you were asked to decide which class did better on the test how would you do it?



Continuous data—from measuring

Discrete data—from counting

12. The heights of the starting 5 members of a junior basketball team are 191 cm, 176 cm, 170 cm, 165 cm, 183 cm.

(a) Find the mean height.

(b) Find the range of height.

13. The defensive line of the school senior football team have masses 86.3 kg, 83.2 kg, 95.5 kg, 79.5 kg, 85.2 kg.

(a) Find the mean mass.

(b) Find the range of masses.

14. A test car covered a measured kilometre in times of 15.2 s, 17.4 s, 15.4 s, and 17.1 s (two runs with the wind and two runs against it).

(a) Find the fastest and slowest rates in km/h.

(b) Find the mean time

(c) Find the mean rate.

15.2 GROUPING DATA TO FIND THE MEAN

EXAMPLE 1. *Group the test marks appearing in the margin to find the mean.*

Test Marks out of 20

12, 15, 10, 5, 10, 11, 12, 7, 8, 11,
13, 14, 11, 8, 10, 9, 11, 11, 10, 10,
14, 9, 12, 13, 6, 9, 12, 13, 8, 12,
11, 12, 9, 11, 8, 11, 11, 10, 9, 6,
10, 13, 19, 18, 17, 18, 18, 17, 16, 20,
16, 18, 17, 19, 7, 9, 7, 10, 10, 11,
10, 8, 12, 9, 13, 15, 19, 13, 15, 16,
18, 11, 16, 14, 12, 15, 12, 14, 11, 9,
11, 8, 14, 12, 13, 10, 7, 11, 8, 15,
13, 6, 9, 9, 10, 12

Table 15-2

Solution When the decision is made to group the data, we must fix the values which are to belong to each class. Since there are 21 possible values, 0 to 20, a grouping of three values per class is convenient.

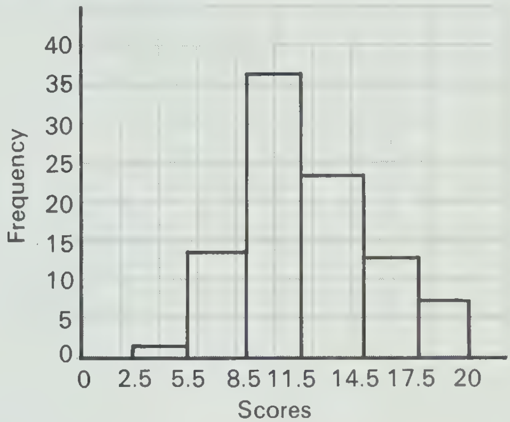
Class values	Tally	Class mid-values x	Frequency f	Frequency × mid-values $x \cdot f$
0–2		1	0	0
3–5	/	4	1	4
6–8	𝍸 𝍸 ////	7	14	98
9–11	𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 /	10	36	360
12–14	𝍸 𝍸 𝍸 𝍸 ////	13	24	312
15–17	𝍸 𝍸 ///	16	13	208
18–20	𝍸 ///	19	8	152
			$n = 96$	$\Sigma x \cdot f = 1134$

When grouping data, the middle value in each class is used to represent all values in that class.

$$\text{The mean } \bar{x} = \frac{\Sigma f \cdot x}{n} = \frac{1134}{96} = 11.8$$

Σ Sigma
A Greek letter indicating "The sum of"

Total the marks as given in the margin and find the average. Has there been a significant loss of accuracy through grouping?



The bar width represents the class values from "boundary value" to "boundary value". The boundary between the class 0-2 and 3-5 is 2.5. The above type of graph is called a *histogram*.

EXAMPLE 2. *A sample of 100 bulbs is tested for life length. Find the mean life length from the frequency distribution table.*

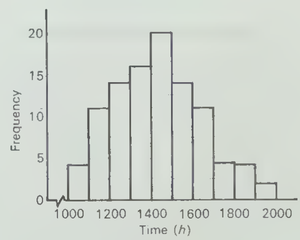
To find class midvalue
Class (1000–1099)

$$\text{Midvalue} = \frac{1000 + 1099}{2} = \frac{2099}{2} = 1049.5$$

Solution

BULB LIFE IN HOURS

Class values	Class mid-values x	Frequency f	$x \cdot f$
0–999		0	0
1000–1099	1049.5	4	4 198.0
1100–1199	1149.5	11	12 644.5
1200–1299	1249.5	14	17 493.0
1300–1399	1349.5	16	21 592.0
1400–1499	1449.5	20	28 990.0
1500–1599	1549.5	13	20 143.5
1600–1699	1649.5	11	18 144.5
1700–1799	1749.5	5	8 747.5
1800–1899	1849.5	4	7 398.0
1900–1999	1949.5	2	3 899.0
Table 15-3		$n = 100$	$\Sigma x \cdot f = 143\ 250$



LIGHT BULB LIFE

(Where classes are large the boundaries are rounded to a whole number.)

$$\text{The mean } \bar{x} = \frac{\Sigma f \cdot x}{n} = \frac{143\ 250}{100} = 1432.5 \text{ h}$$

$\Sigma x \cdot f$ means the sum of all values in the $x \cdot f$ column.

EXERCISE 15-2

B 1. Construct a frequency distribution table and find the mean value from the table of handspans below.

HANDSPANS OF STUDENTS IN CENTIMETRES

Handspan (cm) x	Frequency f	$x \cdot f$
17	1	
18	3	
19	8	
20	12	
21	12	
22	8	
23	4	
24	2	
25	1	

Table 15-4 *Source: Author*

2. Complete the following frequency distribution table in your notebook. Calculate the mean and construct a histogram.

PUSH-UPS BY HIGH SCHOOL BOYS IN 5 MIN

Class values	Class mid-values x	Frequency f	$x \cdot f$
0–4		3	
5–9		5	
10–14		24	
15–19		40	
20–24		21	
25–29		4	
30–34		2	
35–39		1	

Table 15-5 *Source: Author*

3. Construct a frequency distribution table for the number of cars offered for sale by year. Find the mean age. Consider 1974 cars to be one year old. (Table 15-6)

4. Throw three dice 50 times and record the totals of each throw in a frequency distribution with classes 3–4, 5–6, . . . 17–18.

Calculate the mean. Construct a histogram of the results.

Class values	Tally	Class midvalue (x)	Frequency (f)	$x \cdot f$
3–4	///	3.5	(Sample only)	

5. Construct a frequency distribution for the prices of the 1967 cars offered for sale. (Table 15-7.) Calculate the mean price.

CAR ADVERTISEMENT

Cars for sale by year	
Year	**Frequency**
1974	21
1973	101
1972	111
1971	81
1970	87
1969	105
1968	65
1967	50
1966	29
1965	29
1964	13
1963	8
1962	2

Table 15-6
Compiled from a Toronto newspaper

CAR ADVERTISEMENTS

Car prices for 1967 cars		
$ 400	$2200	$ 750
495	450	99
795	525	375
800	600	500
3000	300	600
1200	1300	850
900	350	450
650	700	625
750	625	550
795	1200	1350
300	775	799
1050	1250	

Table 15-7 *Compiled from a Toronto newspaper*

C

6. Table 15-8 gives house prices in $1000's from the advertisements of a daily newspaper. Construct a frequency distribution with classes 25.0–34.9, 35.0–44.9, 45.0–54.9, . . . 125–134.9. Calculate the mean price and construct a histogram. Round off the midvalue to a whole number.

House prices in $1000's				
62.5	50.9	52.0	56.5	53.5
80.0	114.9	49.9	29.9	85.9
45.0	66.9	66.9	67.9	45.9
47.5	53.9	72.9	74.5	39.0
54.5	39.9	37.9	63.9	42.9
109.5	66.5	39.5	39.0	41.5
39.0	45.9	48.0	108.0	69.9
46.9	37.5	38.5	55.9	44.9
43.9	48.0	72.9	39.5	37.9
64.9	74.5	69.9	66.5	68.5
52.5	55.9	54.0	39.5	42.9
69.0	82.5	64.9	53.9	39.9
35.9	74.9	39.5	39.5	62.9
56.9	53.9	54.5	65.5	58.9
99.5	42.9	25.9	110.0	35.9
49.5	82.9	88.5	63.9	62.9
47.9	47.9	51.0	35.9	81.9
75.0	67.9	74.9	49.9	55.9
66.9	60.0	99.5	79.6	65.9
46.5	125.0	56.0	38.9	107.0
54.5	43.9	64.9	49.5	51.0
53.9	44.9	79.9	79.9	72.9
61.9	41.0	87.0	55.9	72.0
57.9	53.5	56.9	45.9	53.9
76.9	85.0	53.9	53.9	52.9

Table 15-8 *Compiled from a Toronto newspaper*

15.3 GROUPING DATA TO FIND THE MEDIAN

EXAMPLE 1. *Group the test marks which appear in the margin and find the median.*

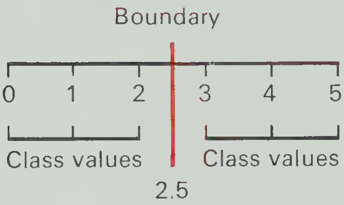
Test Marks out of 20

12, 15, 10, 5, 10, 11, 12, 7, 8, 11,
13, 14, 11, 8, 10, 9, 11, 11, 10, 10,
14, 9, 12, 13, 6, 9, 12, 13, 8, 12,
11, 12, 9, 11, 8, 11, 11, 10, 9, 6,
10, 13, 19, 18, 17, 18, 18, 17, 16, 20,
16, 18, 17, 19, 7, 9, 7, 10, 10, 11,
10, 8, 12, 9, 13, 15, 19, 13, 15, 16,
18, 11, 16, 14, 12, 15, 12, 14, 11, 9,
11, 8, 14, 12, 13, 10, 7, 11, 8, 15,
13, 6, 9, 9, 10, 12

Table 15-2

Solution We shall use the same classes that we used when grouping to find the mean. This time, however, we are interested in class boundaries rather than class midvalues.

Boundary—the number half way between the highest class value in one class and the lowest class value in the next.



Class values	Frequency	Boundary	Cumulative frequency
		0	0
0–2	0		
		2.5	0
3–5	1		
		5.5	1
6–8	14		
		8.5	15
9–11	36		
		11.5	51
12–14	24		
		14.5	75
15–17	13		
		17.5	88
18–20	8		
		20	96

$n = 96$

When calculating the median from boundary values we do not add 1 to n.

Since $n = 96$

$\frac{1}{2} \times 96$

$= 48$

M is the 48th value.

From the cumulative frequency column we see that the 48.5th value will be between 8.5 and 11.5.

We shall use the property of a straight line to interpolate.

$$\text{Slope } AB = \text{Slope } AC$$

$$\frac{48 - 15}{M - 8.5} = \frac{51 - 15}{11.5 - 8.5}$$

$$\frac{33}{M - 8.5} = 12$$

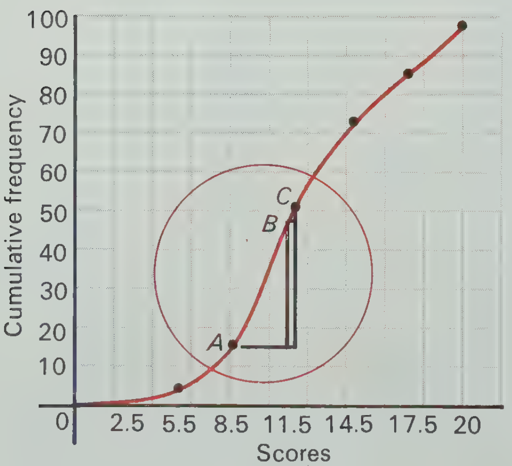
$$12(M - 8.5) = 33$$

$$M - 8.5 = \frac{33}{12}$$

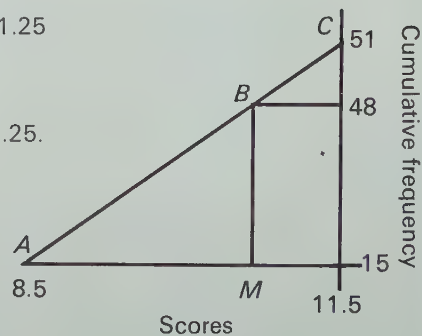
$$M - 8.5 = 2.75$$

$$M = 11.25$$

The median is 11.25.



This graph is an example of a *cumulative frequency polygon*.

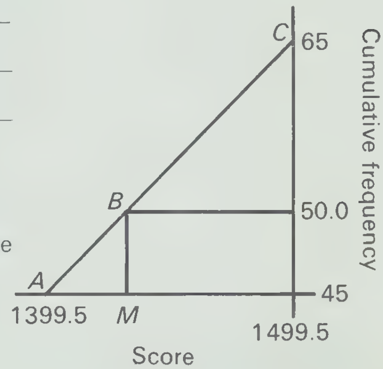


EXAMPLE 2. *Find the median life of light bulbs using Table 15-3.*

Solution

Class values	Frequency	Boundary	Cumulative frequency
0–999	0	999.5	0
1000–1099	4	1099.5	4
1100–1199	11	1199.5	15
1200–1299	14	1299.5	29
1300–1399	16	1399.5	45
1400–1499	20	1499.5	65
1500–1599	13	1599.5	78
1600–1699	11	1699.5	89
1700–1799	5	1799.5	94
1800–1899	4	1899.5	98
1900–1999	2	1999.5	100
	$n = 100$		

$$\frac{100}{2} = 50$$



The median value is the 50th value.

From the cumulative frequency column we note that the median value is between 1399.5 and 1499.5.

$$\text{Slope } AB = \text{Slope } AC$$

$$\frac{50.0 - 45}{M - 1399.5} = \frac{65 - 45}{1499.5 - 1399.5}$$

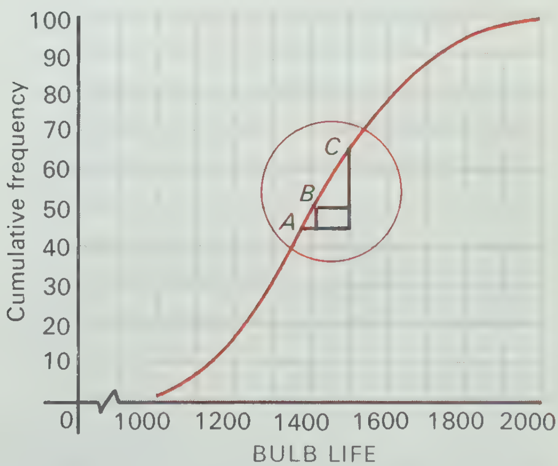
$$\frac{5.0}{M - 1399.5} = \frac{20}{100}$$

$$M - 1399.5 = \frac{100 \times 5.0}{20}$$

$$M = 25.0 + 1399.5$$

$$= 1424.5$$

The median life of a bulb was 1424.5 h.



Note: The horizontal scale has been simplified.

EXERCISE 15-3

B **1.** Complete the table below and calculate the median handspan to the nearest centimetre.

Handspan (cm) x	Frequency f	Cumulative frequency
17	1	
18	3	
19	8	
20	12	
21	12	
22	8	
23	4	
24	2	
25	1	

2. In your notebook complete the following frequency distribution table. Calculate the median and construct a cumulative frequency polygon.

PUSH-UPS BY HIGH SCHOOL BOYS IN 5 MIN

Class values	Frequency	Boundaries	Cumulative frequency
		0	0
0–4	3		
		4.5	3
5–9	5		
		9.5	8
10–14	24		
15–19	40		
20–24	21		
25–29	4		
30–34	2		
35–39	1		

CAR ADVERTISEMENT

Cars for sale by year	
Year	**Frequency**
1974	21
1973	101
1972	111
1971	81
1970	87
1969	105
1968	65
1967	50
1966	29
1965	29
1964	13
1963	8
1962	2

Table 15-6
Compiled from a Toronto newspaper

3. Calculate the median car age from the data shown in Table 15-6. Consider 1974 cars to be one year old.

4. Calculate the median price for 1967 cars from Table 15-7.

CAR ADVERTISEMENTS

Car prices for 1967 cars		
$ 400	$2200	$ 750
495	450	99
795	525	375
800	600	500
3000	300	600
1200	1300	850
900	350	450
650	700	625
750	625	550
795	1200	1350
300	775	799
1050	1250	

Table 15-7 *Compiled from a Toronto newspaper*

B	00.99GROC
B	00.20GROC
B	00.20GROC
B	00.49GROC
B	00.79MEAT
B	00.79MEAT
B	00.79MEAT
B	02.39MEAT
B	00.31 TXBL ITEM
B	00.59GROC
B	00.43PROD
B	00.69GROC
B	00.49PROD
B	00.79PROD
B	00.39PROD
B	00.02 TAX
B	10.35TOTL

5. Find the median price of articles purchased as shown on the cash register slip in the margin. Find the mean price and compare it to the median price. What single item has had the greatest effect on the mean? How does this account for the mean being greater than the median? Do not count the tax as an item.

C 6. Using the real estate data shown in table 15-8, construct a frequency distribution table and calculate the median house price. Construct a cumulative frequency polygon.

House prices in $1000's				
62.5	50.9	52.0	56.5	53.5
80.0	114.9	49.9	29.9	85.9
45.0	66.9	66.9	67.9	45.9
47.5	53.9	72.9	74.5	39.0
54.5	39.9	37.9	63.9	42.9
109.5	66.5	39.5	39.0	41.5
39.0	45.9	48.0	108.0	69.9
46.9	37.5	38.5	55.9	44.9
43.9	48.0	72.9	39.5	37.9
64.9	74.5	69.9	66.5	68.5
52.5	55.9	54.0	39.5	42.9
69.0	82.5	64.9	53.9	39.9
35.9	74.9	39.5	39.5	62.9
56.9	53.9	54.5	65.5	58.9
99.5	42.9	25.9	110.0	35.9
49.5	82.9	88.5	63.9	62.9
47.9	47.9	51.0	35.9	81.9
75.0	67.9	74.9	49.9	55.9
66.9	60.0	99.5	79.6	65.9
46.5	125.0	56.0	38.9	107.0
54.5	43.9	64.9	49.5	51.0
53.9	44.9	79.9	79.9	72.9
61.9	41.0	87.0	55.9	72.0
57.9	53.5	56.9	45.9	53.9
76.9	85.0	53.9	53.9	52.9

Table 15-8 *Compiled from a Toronto newspaper*

15.4 PERCENTILES

In some instances it is more important to know where a score is ranked relative to other scores than to know what percentage the score represents.

Joe and Sally both wrote a mathematics contest. Joe received a score of 33 out of 50, and Sally received a score of 25 out of 50. Joe is in grade 10, and Sally is in grade 9. Who achieved the best results on the test?

From the information given, it is difficult to compare the two results. If we are told that the median mark for all grade 10's writing the test is 32, and the median mark for all grade 9's is 21, we can have a better appreciation of the test scores. Joe is 1 mark above the median for his group, and Sally is 4 marks above the median for hers.

For a more reliable comparison, we might like to know that Joe was as good or better than 52% of the grade 10 students and Sally was as good or better than 56% of the grade 9's. Then we would say that Joe was in the 52nd percentile and Sally was in the 56th percentile.

EXAMPLE 1. *(a) Student A came 25th out of 240 students. What percentile is this student in?*
(b) Student B is in the 25th percentile out of 144 students. What is this student's rank?

Solution
(a) 24 Students did better than Student A. $240 - 24 = 216$. Student A did as well or better than 216 students.

$$\frac{216}{240} \times 100\% = 90\%$$

Student A is in the 90th percentile.

(b) $\frac{25}{100} \times 144 = 36$

Student B did as well or better than 36 students.

$$144 - 36 = 108$$

There are 108 students who did better than Student B. Student B ranks 109th.

EXAMPLE 2. *The frequency distribution table for a set of contest scores is given below.*
(a) What score must a student get to be in the 75th percentile?
(b) In what percentile does a score of 37 place the student?

Evaluate

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}}}$$

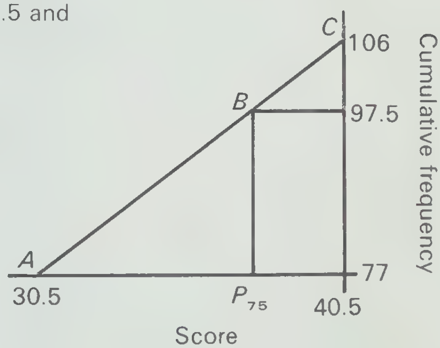
Class values	Frequency	Boundaries	Cumulative frequency
		0	0
0–10	14		
		10.5	14
11–20	25		
		20.5	39
21–30	38		
		30.5	77
31–40	29		
		40.5	106
41–50	18		
		50.5	124
51–60	4		
		60.5	128
61–70	2		
		70	130

Table 15-9

Solution

(a) $\frac{75}{100} \times 130 = 97.5$

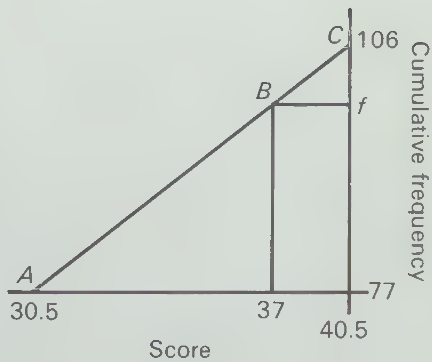
$\therefore$ 75th percentile (P_{75}) corresponds to the 97.5th score. From the cumulative frequency column we see that this score is between 30.5 and 40.5.



$$\text{Slope } AB = \text{Slope } AC$$
$$\frac{97.5 - 77}{P_{75} - 30.5} = \frac{106 - 77}{40.5 - 30.5}$$
$$\frac{20.5}{P_{75} - 30.5} = \frac{29}{10}$$
$$P_{75} - 30.5 = \frac{10 \times 20.5}{29}$$
$$P_{75} - 30.5 = 7.1$$
$$P_{75} = 37.6$$

The 75th percentile score is 37.6.

(b) Find the cumulative frequency corresponding to a score of 37. From the cumulative frequency table we see that this score corresponds to a cumulative frequency between 77 and 106.



$$\text{Slope } AB = \text{Slope } AC$$
$$\frac{f - 77}{37 - 30.5} = \frac{106 - 77}{40.5 - 30.5}$$
$$\frac{f - 77}{6.5} = \frac{29}{10}$$
$$f - 77 = \frac{29 \times 6.5}{10}$$
$$f - 77 = 18.9$$
$$f = 95.9$$

Find the percentile of the 95.9th score.

$$P = \frac{95.9}{130} \times 100\%$$
$$= 73.8\%$$

A score of 37 places a student in the 74th percentile.

EXERCISE 15-4

A

1. Mary-Ann received 74% on her history exam and was in the 80th percentile. Explain the meaning of each score.

2. Tony earned $\frac{35}{50}$ on a mathematics test and came 11th in a class of 25. Find his percentage and his percentile scores.

B

3. Find each of the following scores as a percentage and as a percentile.

	Score	Rank		Score	Rank
(a)	$\frac{15}{25}$	$\frac{13}{22}$	(b)	$\frac{65}{80}$	$\frac{9}{35}$
(c)	$\frac{5}{20}$	$\frac{38}{40}$	(d)	$\frac{148}{250}$	$\frac{40}{80}$
(e)	$\frac{65}{75}$	$\frac{55}{75}$	(f)	$\frac{65}{75}$	$\frac{11}{75}$

4. Find the rank corresponding to each of the following percentile marks (to the nearest whole number).

(a) 82nd percentile out of 45
(b) 64th percentile out of 173
(c) 36th percentile out of 173
(d) 77th percentile out of 26
(e) 47th percentile out of 15
(f) 73rd percentile out of 5750
(g) 8th percentile out of 145
(h) 17th percentile out of 58

5. Three frequently referred-to percentiles are the quartiles. These are:

the first quartile, corresponding to the 25th percentile — $P_{25} = Q_1$ first quartile

the second quartile, corresponding to the 50th percentile — $P_{50} = M$ median

the third quartile, corresponding to the 75th percentile — $P_{75} = Q_3$ third quartile

Since the second quartile is the middle mark, it is usually referred to as the median.

From the following scores select the three quartile scores. 5, 6, 8, 8, 9, 9, 9, 11, 11, 11, 11, 12, 12, 12, 13, 13, 13, 14, 14, 15.

6.

TEST SCORES

Class values	Frequency	Boundary values	Cumulative frequency
0–15	5		
16–30	12		
31–45	16		
46–60	24		
61–75	38		
76–90	42		
91–105	25		
106–120	12		

Table 15-10

(a) Complete the above frequency distribution table in your notebook.
(b) What percentile corresponds to scores of (i) 35 (ii) 74 (iii) 108?

(c) What scores correspond to (i) 10th percentile, (ii) 40th percentile, (iii) 80th percentile?
(d) What scores are below the first quartile?
(e) What scores are above the third quartile?

C **7.** Select a task, such as finding the product of 475×369. Find the time in seconds required to complete the task by 30 subjects.
(a) Construct a frequency distribution table for the times.
(b) Find the three quartile times Q_1, M, Q_3.
(c) Complete the task yourself and find your percentile score.
(d) Construct a histogram of the results.

15.5 MEASURES OF DISPERSION. SEMI-INTERQUARTILE RANGE

In Section 15.1 of this chapter we reviewed the use of the range as a measure of dispersion. The range is easy to calculate and gives valuable information, but it has drawbacks. The range is very sensitive to extreme values as shown by the example in the margin. It is impossible to calculate the range if the data is open-ended, for example if the first class is given as "under 12" or if the last class is given as "over 50".

MARKS ON AN ASSIGNMENT

x	f
25	2
24	3
23	2
21	6
20	9
19	7
17	6
0	1

$R = 25$

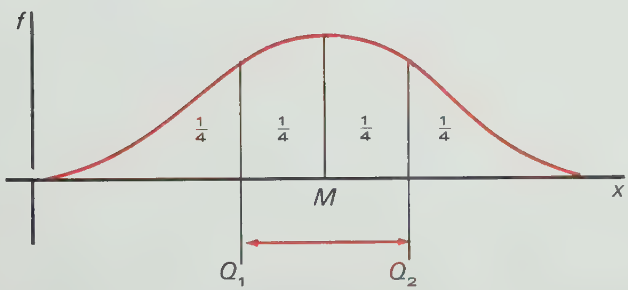
A statistic that overcomes these difficulties is the "interquartile range"

As discussed in question 5 of the previous exercise, the 25th percentile, the 50th percentile and the 75th percentile are given special names. Because they divide the distribution into quarters, they are called quartiles. One-half the values lie between Q_1 and Q_3.
Therefore, the interquartile range $(Q_3 - Q_1)$ gives a range which contains half the values.

$P_{25} = Q_1$ 1st quartile
$P_{50} = Q_2 = M$ median
$P_{75} = Q_3$ 3rd quartile

The semi-interquartile range is one-half the interquartile range:

$$Q = \frac{Q_3 - Q_1}{2}$$



Interquartile range

Boundary	Frequency	Cumulative frequency
0		0
	14	
10.5		14
	25	
20.5		39
	38	
30.5		77
	29	
40.5		106
	18	
50.5		124
	4	
60.5		128
	2	
70		130

From Table 15-9

EXAMPLE 1. *Find the interquartile range and the semi-interquartile range for the contest scores given in the margin.*

Solution

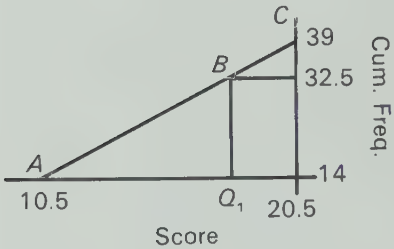
$Q_1 = P_{25}$ $\qquad\qquad$ $Q_3 = P_{75}$

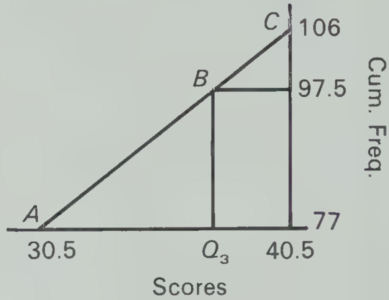
$\frac{25}{100} \times 130 = 32.5$ $\qquad$ $\frac{75}{100} \times 130 = 97.5$

Q_1 corresponds to the 32.5th score. Q_3 corresponds to the 97.5th score.

From the cumulative frequency column:

Q_1 is between 10.5 and 20.5 $\qquad$ Q_3 is between 30.5 and 40.5





$$\text{Slope } AB = \text{Slope } AC$$
$$\frac{32.5 - 14}{Q_1 - 10.5} = \frac{39 - 14}{20.5 - 10.5}$$
$$\frac{18.5}{Q_1 - 10.5} = \frac{25}{10}$$
$$25(Q_1 - 10.5) = 18.5 \times 10$$
$$Q_1 - 10.5 = \frac{185}{25}$$
$$= 7.4$$
$$Q_1 = 7.4 + 10.5$$
$$= 17.9$$

$$\text{Slope } AB = \text{Slope } AC$$
$$\frac{97.5 - 77}{Q_3 - 30.5} = \frac{106 - 77}{40.5 - 30.5}$$
$$\frac{20.5}{Q_3 - 30.5} = \frac{29}{10}$$
$$29(Q_3 - 30.5) = 10 \times 20.5$$
$$Q_3 - 30.5 = \frac{205}{29}$$
$$= 7.1$$
$$Q_3 = 7.1 + 30.5$$
$$= 37.6$$

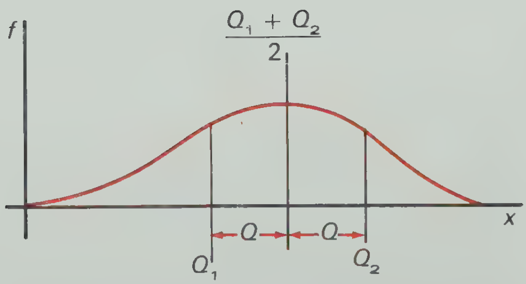
$$\text{Interquartile range} = Q_3 - Q_1$$
$$= 37.6 - 17.9$$
$$= 19.7$$

$$\text{Semi-interquartile range} = \frac{Q_3 - Q_1}{2}$$
$$= \frac{19.7}{2}$$
$$\doteq 9.9$$

Half the scores are within a range of 19.7.

Note: $\frac{Q_3 + Q_1}{2} = \frac{37.6 + 17.9}{2}$

$$= \frac{55.5}{2}$$
$$= 27.8$$



Q − semi-interquartile range

If the distribution is symmetrical about the median, $M = 27.8$, half the scores lie within 9.9 units of 27.8, from 17.9 to 37.7.

EXERCISE 15-5

A **1.** In a sample of 100 metal bolt washers manufactured to a thickness of 2.0 mm, quality control statistics showed $1.80 \leq x \leq 2.95$, where x is the thickness in mm.

$$Q_1 = 1.94 \text{ mm}$$

$$Q_2 = 2.08 \text{ mm}$$

(a) How many values lie between 1.80 mm and 2.95 mm, inclusive?
(b) What is the range of thickness?
(c) How many values lie between 1.94 mm and 2.08 mm?
(d) What is the interquartile range?
(e) What is the semi-interquartile range?
(f) Estimate the median thickness.

B **2.** For the handspans shown in the table in the margin
(a) Calculate the range.
(b) Calculate the semi-interquartile range.
(c) Estimate the median from the values for Q_1 and Q_3.

Handspan (cm)	Frequency
17	1
18	3
19	8
20	12
21	12
22	8
23	4
24	2
25	1

From Table 15-4

3. Find the semi-interquartile range for the test scores in the margin.

TEST SCORES

Class values	Frequency
0–15	5
16–30	12
31–45	16
46–60	24
61–75	38
76–90	42
91–105	25
106–120	12

From Table 15-10

4. At an intersection with traffic lights, a count was made of the number of cars that pass through during the interval of one green light. (Table 15-11.)
(a) Find the median (M) number of the cars to pass the intersection in one light.
(b) Find the semi-interquartile range $\dfrac{Q_3 - Q_1}{2}$.
(c) Calculate $\dfrac{Q_1 + Q_3}{2}$ and compare this value to M. What does this tell you about the symmetry of the distribution?

CARS PASSING AN INTERSECTION ON ONE LIGHT CHANGE

No. of cars	Frequency
0–2	7
3–5	17
6–8	24
9–11	12
12–14	8
15–17	3
18–over	3

Table 15-11

5. Find the semi-interquartile range for used car prices in Table 15-7.

6. (a) Find the range of house prices in Table 15-8.
(b) Find the semi-interquartile range.
(c) What is the advantage of the semi-interquartile range over the range?

C 7. Using the data you collected for your task in Exercise 15-4, Question 7, determine the range and semi-interquartile range.

15.6 MEASURES OF DISPERSION
Standard Deviation

One of the most often used measures of dispersion is the *standard deviation* of the values from the mean. Consider the set of test marks in the following frequency distribution.

Class values	Class midvalues x	Frequency f	$f \cdot x$	$x - \bar{x}$	$(x - \bar{x})^2$	$f \cdot (x - \bar{x})^2$
15–19	17	1	17	−15	225	225
20–24	22	3	66	−10	100	300
25–29	27	4	108	−5	25	100
30–34	32	8	256	0	0	0
35–39	37	6	222	5	25	150
40–44	42	2	84	10	100	200
45–49	47	1	47	15	225	225
		$n = 25$	$\Sigma f \cdot x = 800$	$\Sigma(x - \bar{x}) = 0$		$\Sigma f \cdot (x - \bar{x})^2 = 1200$
	The midvalue is used to represent all values in the class	$\bar{x} = \frac{800}{25}$ $= 32$ The mean is found as in section 15.1.		The difference between each midvalue and the mean calculated. Note the sum of the difference $(x - \bar{x}) = 0$.	To make the sum a non-zero number and to make the measure more sensitive to larger differences, the differences are squared.	Each $(x - \bar{x})^2$ is multiplied by the frequency of the class it represents and the average is found.

$$\frac{\Sigma f \cdot (x - \bar{x})^2}{n} = \frac{1200}{25}$$
$$= 48$$

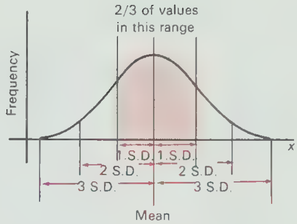
Since all values were squared before finding the average dispersion, the square root is is now taken.

$$\sqrt{\frac{\Sigma f \cdot (x - \bar{x})^2}{n}} = \sqrt{48}$$
$$= 6.9$$

$$\boxed{\text{S.D.} = \sqrt{\frac{\Sigma f \cdot (x - \bar{x})^2}{n}}}$$

There are two important properties of the standard deviation:

1. Approximately $\frac{2}{3}$ of all values lie within one standard deviation of the mean.
2. Practically all values lie within three standard deviations of the mean (99.7% for a normal distribution).

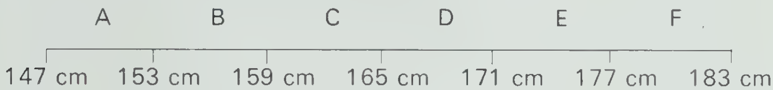


68.3% will be within one standard deviation of the mean.
95.5% will be within two standard deviations of the mean.
99.7% will be within three standard deviations of the mean.

EXERCISE 15-6

A 1. A sample of 1000 roller bearings has a mean diameter of 25.00 mm with a standard deviation of 0.02 mm.
(a) Approximately how many roller bearings have a diameter between 24.98 mm and 25.02 mm?
(b) Between what limits would practically all (99.7%) the values of the diameters lie?

2. The average height in a class of 50 boys is 165 cm with a standard deviation of 6 cm. If the heights are normally distributed, how many boys should be in each range illustrated below.

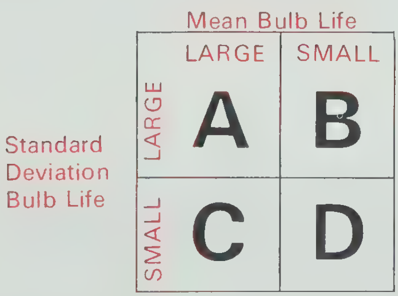


3. What difficulty would be encountered in finding the standard deviation of the data in table 15-11?

B 4. The class marks on a mathematics test are given in the margin. Find the mean and the standard deviation.

5. Find the standard deviation of handspans given in table 15-4.

6. A metal pin must be machined to have a diameter between limits of 19.4 mm and 20.6 mm.
(a) Assuming the lathe was as likely to machine too large as too small, what mean diameter should be machined?
(b) If it is acceptable to have 0.3% defective parts, what maximum standard deviation of the diameters is acceptable?



7. (a) Find the standard deviation of the bulb life statistics given in the margin.
(b) Batches of bulbs were found to have the following characteristics:
long mean life and large standard deviation
short mean life and large standard deviation
long mean life and small standard deviation
short mean life and small standard deviation.
Grade each set of characteristics "Best", "Poor", or "Worst".

CLASS MARKS ON A TEST OUT OF 25

x	f
25	1
24	1
23	2
22	4
21	5
20	4
19	5
18	4
17	3
16	3
15	1
14	2
13	1
12	2
11	0
10	1

BULB LIFE IN HOURS

Class midvalues x	Frequency f
1049.5	4
1149.5	11
1249.5	14
1349.5	16
1449.5	20
1549.5	13
1649.5	11
1749.5	5
1849.5	4
1949.5	2

$\bar{x} = 1432.5$ h

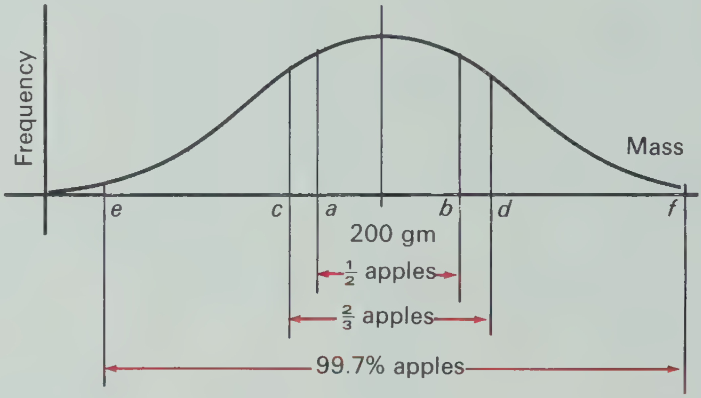
From Table 15-3

Investment yields

Banks		Rate	Price	Yield
Canadienne		.96	19.50	4.92
Imperial Commerce		1.12	31.00	3.61
Montreal		.96	20.75	4.63
Nova Scotia		1.20	40.50	2.96
Royal		1.08	38.25	2.82
Toronto Dominion		1.12	40.00	2.80
PREFERRED				
Alcan Aluminum 4¼%		1.70	24.25	7.01
Argus		2.50	31.00	8.06
Bell Canada		3.20	45.25	7.07
Canada Cement		1.30	17.00	7.65
C.P. Inv. 4¾%		.95	35.25	2.70
Cdn. Pacific 7¼% A		.72½	10.50	6.90
Carling O'Keefe		2.20	23.75	9.26
Domtar		1.00	13.50	7.41
GSW Ltd. C		5.00	84.00	5.95
Hudson's Bay Oil A 5%		2.50	48.50	5.15
Jannock Corp. A		1.20	13.00	9.23
Loblaw Companies		2.40	27.75	8.65
Maple Leaf Mills		5.50	100.00	5.50
North. & Cen. Gas B		1.50	24.50	6.12
Power Corp. 4¾%		2.37	30.50	7.77
Silverwoods A		.80	12.50	6.40
Westburne 8% A		2.00	36.50	5.48
Weston		4.50	59.25	7.59
COMMON				
Abitibi		.40	14.25	2.81
Alberta Gas Trunk A		.42	13.75	3.05
Alcan		1.00	33.50	2.99
Algoma Central		.60	13.50	4.44
Algoma Steel		1.00	25.25	3.96
Bell Canada		3.12	44.25	7.05
Brascan		1.00	17.25	5.80
B.C. Forest		.80	19.75	4.05
Calgary Power		1.20	24.75	4.85
Canada Cement		.60	14.00	4.29
Canada Malting A		1.40	29.50	4.75
Canada Packers		.88	24.25	3.63
Canada Perm. Mtge.		.88	20.00	4.40
Cdn. Cablesystems		.28	16.00	1.75
Cdn. Industries		1.00	19.75	5.06
Cdn. Pacific Ltd.		.77	17.00	4.53
Canron Ltd.		1.00	23.00	4.35
Cominco Ltd.	1.20+	.15	33.00	4.09
Consumers' Gas		.88	17.75	4.96
Corby's Vtg.	1.00+	.50	21.75	6.90
Crain, R. L.		.40	8.50	4.71
Dist. Seagrams		.80	45.00	1.78
Dom. Bridge	1.40+	.30	42.25	4.02
Dom. Foundries		1.20	32.75	3.66
Dom. Stores		.72	14.50	4.97
Domtar	1.00+	.30	27.00	4.81
Dom. Textile		.40	10.25	3.90
DuPont		1.00	28.00	3.57
Ford of Canada	3.40+	1.25	84.00	4.35
Genstar		.90	19.25	4.68
Gulf Oil Canada	.60+	.10	35.75	1.96
Hayes-Dana		.24	6.75	3.56
Hudson's Bay Co.		.60	18.00	3.33
IAC Limited		.96	19.50	4.92
Imasco Ltd.		1.00	31.50	3.17
Imperial Oil	.80+	.10	39.25	2.29
Int. Nickel	1.20+	.15	38.00	3.55
Interprov. Pipe		1.20	21.25	5.65
Jannock Corp.		.40	6.75	5.98
Labatts		.92	23.00	4.00
Loblaw B		.38	6.50	5.85
Maclean-Hunter A		.30	9.00	3.33
MacMillan Bloedel		1.50	32.00	4.69
Maple Leaf Mills		1.40	23.75	5.89
Molson Companies A		.80	23.25	3.44
Moore Corp.		.80	51.50	1.55
Noranda A		1.60	53.00	3.02
No. & Central Gas		.61	12.50	4.80
Oshawa Group		.34	7.50	4.53
Pac. Petroleums		.60	33.00	1.82
Power Corp.		.30	12.00	2.50
Simpsons		.20	9.00	2.22
Southam		.80	28.50	2.81
Std. Broadcasting	.40+	.10	8.25	6.06
Standard Industries		.50	9.00	5.56
Steel of Canada		1.40	34.00	4.12
Thomson Newspapers		.22	13.00	1.69
Traders Group A		1.00	17.00	5.88
TransCan PipeLines		1.00	33.75	2.96
Trans Mountain	1.20+	.07½	16.00	7.97
Union Carbide		.65	18.00	3.61
Union Gas		.64	10.00	6.40
Walkers G & W	1.40+	.25	53.00	3.11
Weston		1.12	22.75	4.92
MINING				
Bethlehem	.60+	.10	14.50	4.83
Camflo	.40+	.10	18.50	2.70
C'bell Red Lake	.50+	.15	78.00	0.83
Cassiar		.60	11.25	5.33
Craigmont		.60	6.00	10.00
Denison		1.40	50.00	2.80
Dome Mines	1.00+	.50	153.00	0.98
Falconbridge Copper		.80	12.50	6.40
Gt Yellowknife	.40+	.20	19.25	3.12
Hollinger		1.60	41.25	3.88
H'son Bay M&S	1.60+	.50	26.25	8.00
Iron Bay Trust		.20	4.15	4.82
Kerr Addison	.40+	.20	14.50	4.14
Labrador		1.80	39.50	4.56
Madeleine		.80	5.25	15.24
Mattagami Lake		1.40	34.50	4.06

C **8.** Find the standard deviation of the house prices listed in the advertisements in Table 15-8.

9. A sample of 100 apples has a mean mass per apple of 200 g with a standard deviation of 10.0 g and a semi-interquartile range of 6.8 g. Find *a*, *b*, *c*, *d*, *e*, and *f*.



REVIEW EXERCISE

A **1.** Give three measures of central tendency and an example situation where each would be useful.

2. Give three measures of dispersion and an example situation where each would be useful.

3. Explain the difference between a percentage mark and a percentile mark. How do the two complement each other in some situations?

B **4.** From the listing of investment yields given in the margin.

(a) Group the data into classes, construct a frequency distribution, and calculate the mean and the standard deviation of investment yields. Use classes 0.01–2.00, 2.01–4.00, . . . 14.01–16.00.

Class values	Class midvalues x	Frequency f	$x \cdot f$	$x - \bar{x}$	$(x - \bar{x})^2$	$f \cdot (x - \bar{x})^2$
0.01–2.00	1.00	8	8.00	−5.6	31.36	250.88

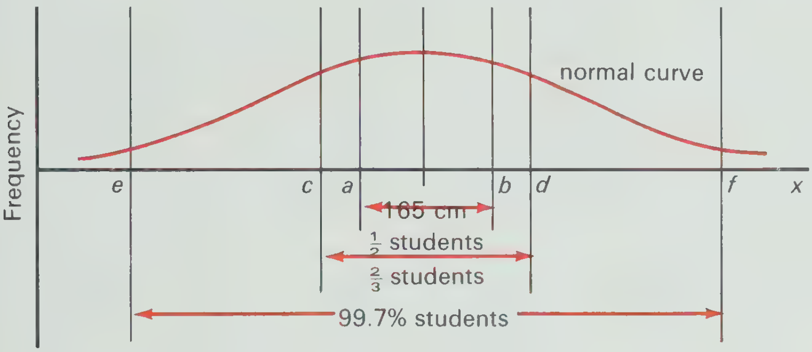
(b) Using the same classes as in **4.** (a), construct a frequency distribution and calculate the median and the semi-interquartile range.

Class values	Frequency	Boundary values	Cumulative frequency

(c) Construct a histogram of the investment yields. Draw a vertical line on the graph at each quartile mark.

5. (a) On a mechanical aptitude test, Richard received a mark which placed him in the 64th percentile of his group of 80. How many students received a mark lower than Richard's?
(b) If Alice came 60th out of 80 students, what percentile was she in?

6. A group of 78 students has a mean height of 165 cm with standard deviation of 6.5 cm. The semi-interquartile range is 4.7 cm. Find *a, b, c, d, e, f*.



7. A taxi driver carried 33 fares over one weekend. The distances involved for the trips are given in Table 15-13.
(a) Construct a frequency distribution using a class interval of 0.5 km.
(b) Calculate the median trip length by interpolation over the interval.
(c) Find the actual median by determining the middle value.

DISTANCE OF TAXI TRIPS

2.8 km	2.4 km
1.0	1.2
2.3	3.4
1.6	3.5
2.1	2.7
1.4	1.5
2.4	0.9
0.3	2.5
1.8	1.4
1.3	2.0
2.4	1.3
0.3	1.6
1.2	1.1
1.6	1.9
1.4	2.1
0.8	4.2
1.7	

Table 15-12

C *Experiments*

8. Toss six coins at once 50 times and record the number of heads turning up on each throw.
(a) Find the mean number of heads to turn up.
(b) Find the standard deviation of the number of heads.
(c) On a bet you win if the number of heads turning up lies within one standard deviation of the mean.
How many times out of 100 would you expect to win?

No. of Heads	0	1	2	3	4	5	6
Frequency							

9. Determine the age in months of each student in your math class. Find the mean age and the median age. What is the range of ages?

10. Draw a chalk line on the board between 0.5 m and 1.0 m in length. Ask each student in the class to estimate the length and write his answer on a piece of paper. Find the semi-interquartile range of the estimates.

ANSWERS

REVIEW AND PREVIEW TO CHAPTER 1

Exercise 1

1. (a) 12 (b) 6 (c) 15 (d) 21 (e) 6 (f) 12
   (g) 30 (h) 4 (i) 5 (j) 4 (k) 2600 (l) 41 000
   (m) 2.4 (n) 0.63 (o) 30 (p) 3 (q) 160 (r) 2500
   (s) 8 (t) $\frac{4}{21}$ (u) $\frac{3}{4}$ (v) 16 (w) 10 (x) 8
   (y) $\frac{2}{5}$ (z) 3

Exercise 2

1. (a) 27 504 (b) 12 177 (c) 18 018 (d) 23 672
   (e) 645 498 (f) 163 218 (g) 100 878 (h) 184 507
   (i) 27 132 (j) 21 774 (k) 145 008 (l) 29 593
   (m) 1 493 558 (n) 128 874 006 (o) 145 268 242 (p) 183 315
2. (a) 68 (b) 45 (c) 459 (d) 710
   (e) 62 (f) 81 (g) 756 (h) 82
   (i) 177 *R*: 19 (j) 1082 *R*: 15 (k) 61 *R*: 82 (l) 31 *R*: 733
   (m) 261 *R*: 52 (n) 140 *R*: 49 (o) 53 *R*: 157 (p) 235 *R*: 27

Exercise 3

1. (a) $5x$ (b) $5a$ (c) $12m + a$ (d) $10x + 2y$
   (e) $8a - 7b$ (f) $-x - y$ (g) $13ab + m$ (h) $17 - 4a + 7b$
   (i) $13s - 2t$ (j) $-7a - xy$ (k) $10ab + 13bc$ (l) $2d - 13$
   (m) $11 - 9e + 3ef$ (n) $-5x^2 + 5x - 6$ (o) $2a^2 + 6ab$
   (p) $11x^3 - 3x^2 - 7x$ (q) $19m^2n + 3mn + 7mn^2$
   (r) $4abc - 8ab - 4a + 5$ (s) $8ab - bc + 3a - 7$
2. (a) $2a + 8$ (b) $3a - 6$ (c) $4x - 12$ (d) $2a - 2b$
   (e) $7x + 7y$ (f) $16x + 8$ (g) $2a + 4b + 2c$ (h) $6x - 12y + 42$
   (i) $-2x + 8$ (j) $-6a + 12b$ (k) $-15a + 10b - 5c$
   (l) $-x - 5$ (m) $10 - 30x + 20y$ (n) $14 + 21x - 28y$

Display 1

1. 93.81
2. 76.01944076
3. 73.30281752
4. 33.41572212
5. 2703.020171
6. 768.1456
7. 29.9077
8. 123.8544
9. 10.359346
10. 0.6672768879

CHAPTER 1

Exercise 1-3

1. 4.9 cm 2. $48.00 3. (a) 920 km (b) 3 h 4. 600 N 5. $33.35

Exercise 1-4

1. $35.92 2. $13.35 3. $227 4. $143.85
5. (a) Publisher B, $17.50 (b) 100 posters
6. (b) $16, $16 (c) $26 (d) $32

REVIEW AND PREVIEW TO CHAPTER 2

Exercise 1

1. (a) 3.188 (b) 4.150 (c) 10.22 (d) 19.918 (e) 9.172
2. (a) 3.924 (b) 5.573 (c) 0.498 (d) 0.524 (e) 1.019
3. (a) 0.3358 (b) 0.035 (c) 1.8002 (d) 0.6041 (e) 2.681 34
4. (a) 4.6 (b) 463 (c) 2.002 (d) 0.76 (e) 7.04

Exercise 2

1. a^{11} 2. b^{15} 3. $8m^{10}$ 4. $6x^3y^2$ 5. $-24m^3n^2$ 6. $-18x^4y^5$ 7. $16a^2b^5$
8. a^2 9. x^2 10. $2m$ 11. $4a$ 12. $-4ab^2$ 13. 5 14. $2ab^2$

Exercise 3

1. 28 2. 24 3. 36 4. 13 5. 19 6. -21 7. 96 8. -102 9. -2
10. 19 11. -49 12. 288 13. -1749 14. 22 15. 20 16. -13
17. -571

Display 2

1. 4.304597701
2. 1.569661591
3. 0.0196233679
4. 4.487731821
5. 0.080375
6. 0.4287886965
7. 0.0238095238
8. 0.0143929949
9. 0.7231949153
10. 0.37373737

CHAPTER 2

Exercise 2-1

1. (a) $6x + 2$ (b) $8a - 12$ (c) $10a + 15b$ (d) $-2x + 14$
(e) $-3x + 3y$ (f) $-2x + 4$ (g) $-21y - 14$ (h) $2x^2 + 4x$
(i) $-2x^2 + 2xy$
2. (a) $3x - 3y - 3t$ (b) $-8a + 12b + 4c$ (c) $-2p + 3q + 6$
(d) $2ax - 2ay - 2at$ (e) $2x^3 - 6x^2 - 6x$ (f) $a^3b - ab^2 - ab$
3. (a) $5a$ (b) $2y + 2$ (c) $2x + 16$ (d) $14b - 1$
(e) $a - 15$ (f) $5x - 2$ (g) $-2x^2 - x - 11$
(h) $a - 2b + 6c$ (i) $-3a^2 - 8a + 10$ (j) $6x^2 + x + 11$
4. (a) $-7x$ (b) $7a^2 - 7ab$ (c) $-6b^2 + 10b - 4$
(d) $-3m^2 + mn + 2n^2$ (e) $x^2 + 11xy - 4x$ (f) $8a^2 - 3a + 7$
(g) $x^2 - 2x + 2$ (h) $4t^2 + t - 2$ (i) $-6a + 2b - 1$ (j) $10t + 3$
5. (a) $8x - 12$ (b) $18a - 42$ (c) $2x^2 + 3x$ (d) $15a^2 - 12a$
(e) $4m - 12$ (f) $12m^2 + 16m$ (g) $5x^2 - 2x$ (h) $8x^2 + 24x$
6. (a) $13m + 58$ (b) $-5x - 13$ (c) $-17a + 8$ (d) $24b + 2$
(e) $27b^2 - 2b + 10$
7. (a) $-2x^2y - xy^2 - 3x$ (b) $-2a^3b$ (c) $-2m^2n - 14mn^2$
(d) $-x - 5y$ (e) $21y$

Exercise 2-2

1. (a) $x^2 + 5x + 6$ (b) $a^2 + 4a - 21$ (c) $b^2 - 9b + 14$
(d) $y^2 - 5y + 6$ (e) $m^2 - 1$ (f) $t^2 + 5t + 6$
(g) $c^2 - 8c + 12$ (h) $x^2 - 9x + 20$ (i) $m^2 + 13m + 42$
(j) $2x^2 + 5x - 3$ (k) $6b^2 + b - 2$ (l) $25a^2 - 1$
2. (a) $x^2 + 14x + 49$ (b) $m^2 - 4m + 4$ (c) $a^2 + 6a + 9$
(d) $4x^2 + 4x + 1$ (e) $16y^2 - 24y + 9$ (f) $1 + 6x + 9x^2$
(g) $4m^2 + 20m + 25$ (h) $1 - 10x + 25x^2$ (i) $4r^2 - 28r + 49$
3. (a) $8a^2 - 2ax - 3x^2$ (b) $x^2 + xy - 6y^2$ (c) $2a^2 + ab - b^2$
(d) $9m^2 - 4b^2$ (e) $x^2 - 4xy + 4y^2$ (f) $4a^2 + 4ab + b^2$
(g) $14d^2 - 15dg - 9g^2$ (h) $4x^2 - y^2$ (i) $36a^2 - 84ab + 49b^2$
(j) $2m^2 + 7m - 4$ (k) $-6x^2 - 13x + 5$ (l) $20 - 43x + 21x^2$
4. (a) $x^2 + 6x - 7$ (b) $2a^2 - 5a - 7$ (c) $4x^2 + 17x + 15$
(d) $5x^2 + x - 4$
5. (a) $x^3 + 3x^2 + 3x + 1$ (b) $a^3 + 5a^2 + 5a - 2$ (c) $b^3 - 2b - 1$
(d) $y^3 - y^2 - 10y - 8$ (e) $2x^3 + 3x^2 - x + 2$ (f) $m^3 + 3m^2 + m - 1$
(g) $x^4 - x^2 - 2x - 1$ (h) $2x^4 + 3x^3 - x^2 + 7x - 3$
(i) $a^4 - 2a^3 - 18a^2 - 41a - 28$ (j) $6y^4 - 7y^3 - 17y^2 - 12y - 12$

Exercise 2-3

1. (a) $x^2 + 7x - 2$ (b) $m^2 + m - 12$ (c) $6x^2 + 6x$ (d) $17a^2 + 5a + 5$
2. (a) $3x^2 - 2x - 26$ (b) $3a^2 - 13a + 30$ (c) $2m^2 + 8m - 8$ (d) $10x^2 + 17x$ (e) $2x^2 + 2x + 13$ (f) $a^2 - 14a + 17$
3. (a) $-a^2 - 30a - 80$ (b) $-t^2 + 19t - 18$ (c) $10m^2 + 24m - 17$ (d) $-2a^2 + 19a + 30$ (e) $x^2 - x - 4$ (f) $-11m^2 + 10m - 2$
4. (a) $-6m^2 - 6m + 2$ (b) $-12x^2 + 7x - 59$ (c) $-23x^2 + 30x - 58$ (d) $-8t^2 - 2t + 6$ (e) $-m^2 + 13m + 5$
5. (a) $x^3 + 7x^2 + 12x + 3$ (b) $-9a^2 + 2a - 5$ (c) $4y^3 - 9y^2 + 24y + 17$ (d) $-12a^3 + 13a^2 + 2a + 7$

Exercise 2-4

1. (a) $5b$ (b) $4c$ (c) $-5y$ (d) $3c$ (e) -5 (f) $2a$ (g) $-6b^2$ (h) $8a$ (i) -2
2. (a) $x + m$ (b) $3a - b$ (c) $a^2 - 1$ (d) $-4ab + b$ (e) $3mt - m$ (f) $ab - 2a + 3$
3. (a) $4b - 2c$ (b) $5c - 1$ (c) $2x - 1$ (d) $-8x - 2$ (e) $3 - 2a + b$ (f) $2n + 1 + 3t$
4. (a) $-10a + 5b$ (b) $-6x^2 - 4x + 1$ (c) $-1 + 2a - 3a^2$ (d) $-2m^2n + n^3$ (e) $-24a^2 + 12ab - a$ (f) $1 - 4a^2b + 3ab^2$
5. (a) $16x$ (b) $8b$ (c) $2a$ (d) $6a$ (e) $7x$ (f) $14ab$
6. (a) $3a$ (b) $10x - 10y$

Exercise 2-5

1. (a) $5x^3 + 2x^2 + 4x - 3$ (b) $7a^4 + 6a^3 - 3a^2 + 5a$ (c) $-7b^2 - 4b + 1$ (d) $8m^4 + 7m^2 - 3m + 1$ (e) $4t^4 - 3t^2 - 5t - 6$ (f) $-7x^6 + 6x^4 + 3x^2 - 2x - 4$
2. (a) $x + 5$ (b) $a - 4$, R: -24 (c) $m + 4$ (d) $x + 2$, R: 1
3. (a) $x^2 - 2x + 1$ (b) $a^2 + 2a - 2$, R: 20 (c) $2b^2 - 3b + 6$, R: -6 (d) $6c^2 + 5c - 1$, R: 1 (e) $4a + 16$, R: 110 (f) $3y^3 + 2y^2 - 3y - 1$, R: 9
4. (a) $a + 4$ (b) $2x^2 + 3x - 3$, R: -27 (c) $r^2 + r + 1$ (d) $6m^3 - 3m^2 - m - 2$, R: 7
5. (a) $2x^4 + 2x^3 + 3x^2 - 2x - 2$ (b) $3y^2 - 2y - 5$ (c) $x^2 + 2x + 1$ (d) $2x^2 - 3xy + y^2$
6. $x^3 - 2x^2 - 5x + 1$ 7. $2x^2 - 9$

Exercise 2-6

1. (a) 17 (b) 10 (c) 16 (d) -6 (e) -12 (f) -8 (g) 5 (h) 20 (i) -4 (j) 3 (k) -4 (l) 0
2. (a) 3 (b) 3 (c) 1 (d) 4 (e) 3 (f) 3 (g) 1 (h) 0 (i) 2
3. (a) yes (b) no (c) yes (d) yes (e) no (f) no (g) yes (h) no
4. (a) 5 (b) 2 (c) -1 (d) 4 (e) 1 (f) 2 (g) 3 (h) 6 (i) 4 (j) 2
5. (a) 6 (b) 9 (c) 3 (d) 3 (e) 2 (f) $\frac{1}{2}$ (g) 0 (h) $\frac{7}{6}$ (i) 2 (j) 5
6. (a) 2 (b) -8 (c) -11 (d) 5 (e) -1 (f) 2 (g) 2 (h) 1
7. (a) 1.2 (b) -1.76 (c) 1.11 (d) 4.79 (e) -3.14 (f) -4.73
8. (a) $7 - b$ (b) $m + t$ (c) $d - t$ (d) $\frac{m}{a}$ (e) $-\frac{a}{2}$ (f) $\frac{a + m}{2}$ (g) $\frac{m - t}{b}$ (h) $\frac{m + t}{a}$ (i) $\frac{a + 3}{2b}$

Exercise 2-7

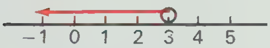
1. (a) 30 (b) 8 (c) 9 (d) -30 (e) $\frac{8}{3}$ (f) 5
2. (a) 12 (b) $\frac{9}{4}$ (c) $-\frac{4}{5}$ (d) $-\frac{20}{7}$ (e) 36 (f) $\frac{15}{2}$
   (g) $-\frac{16}{9}$ (h) $\frac{7}{19}$
3. (a) -7 (b) 6 (c) $\frac{17}{2}$ (d) 2 (e) $-\frac{1}{3}$ (f) 26
   (g) -5 (h) $\frac{36}{17}$
4. (a) 38 (b) -31 (c) -3 (d) 3 (e) 14 (f) $\frac{38}{5}$

Exercise 2-8

1. (a) 2 (b) $\frac{4}{3}$ (c) -8 (d) 1 (e) -1 (f) -1
2. (a) -3 (b) 1 (c) 3 (d) $\frac{7}{2}$ (e) -5 (f) -24
3. (a) 3 (b) -5 (c) 3 (d) 3

Exercise 2-9

1. (a) $x < 3$ (b) $a > 8$ (c) $b \leqq 10$ (d) $m \geqq 4$
   (e) $d < 1$ (f) $y \leqq -8$ (g) $x < -7$ (h) $m \geqq 0$
   (i) $x > 3$ (j) $m \leqq -2$ (k) $b > -3$ (l) $x < -2$
   (m) $a > 5$ (n) $x > -10$ (o) $b \leqq -2$ (p) $x \leqq -10$
   (q) $m < -12$ (r) $t < 9$ (s) $x > 18$ (t) $a \leqq -16$
   (u) $m \geqq -12$ (v) $b > -16$ (w) $c \geqq 3$ (x) $a < 28$
2. (a) $x \geqq 2$ (b) $x < -3$ (c) $x > 0$ (d) $x \leqq 3$
   (e) $x < 6$ (f) $x \geqq -6$ (g) $x \leqq 0$ (h) $x > 3$

3. (a)


(b)


(c)


(d)


(e)


(f)


(g)


(h)

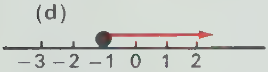

(i)


(j)


4. (a)

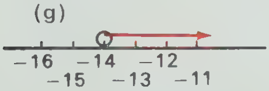

(b)

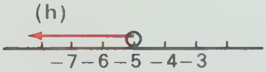

(c)


(d)


(e)

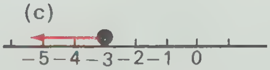

(f)

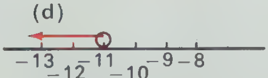

(g)


(h)


5. (a)


(b)


(c)


(d)


(e)


(f)


(g)


(h)


6. (a)

(b)

(c)

(d)

(e)

(f)

(g)

(h)

Exercise 2-10

1. (a) 100 (b) 180 (c) 3 (d) 10 (e) 60 (f) 58 (g) 4 (h) 2
2. (a) 121.44 (b) 11.0 (c) 32 (d) 2 (e) 7.5 (f) 7.1
3. 69 m 4. 16 cm 5. $3.25
6. (a) $t = \frac{P}{4}$ (b) $w = \frac{V}{lh}$ (c) $V = \frac{d}{t}$ (d) $I = \frac{V}{R}$ (e) $R = \frac{pv}{t}$
   (f) $l = \frac{P - 2w}{2}$ (g) $a = \frac{2s}{n} - l$ (h) $a = \frac{2(s - ut)}{t^2}$ (i) $r = \frac{A - p}{pt}$

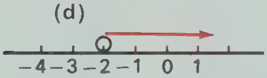
Review Exercise

1. (a) $2c$ (b) $-4a$ (c) $4mn$ (d) $-10b^2$
2. (a) 7 (b) 12 (c) −5 (d) 3 (e) −7 (f) 3
   (g) 8 (h) −50 (i) 5
3. (a) $4x^2 + 12x + 28$ (b) $-2a + 4b$ (c) $-3a + 1$ (d) $2x^3 - 2x^2$
   (e) $-6m^2 + 6m - 3$ (f) $-2x^2 + 2xy$ (g) $-12a - 8b$ (h) $-6a^2b + 9ab^2$
   (i) $3x^2 + 2x - 7$
4. (a) $-a - 33$ (b) $x - 13$ (c) $-8a + 14b - 4c$ (d) $5a^2 - 4a$
   (e) $2m^2 + 4m$ (f) $x^2 - 5xy$
5. (a) $a^2 - a - 6$ (b) $3a^2 + 5a - 2$ (c) $-3x^2 - 2x + 1$ (d) $4x^2 - 4x + 1$
   (e) $x^2 - 4x - 21$ (f) $6x^2 + 7x - 20$ (g) $4b^2 - 12ab + 9a^2$ (h) $1 - 16x^2$
6. (a) $x^2 + x + 8$ (b) $a^2 - 7a - 14$ (c) $b^2 + 10b + 16$ (d) $-3x^2 + x$
   (e) $14t^2 + 15t - 11$ (f) $-2a^2 + 4ab + 7b^2$
7. (a) $x + 3$ (b) $x^2 - 2x$ (c) $5a - 3c$ (d) $-10a + 6a^2$
   (e) $-m + 2n - 3t$ (f) $2x^2 + x - 1$ (g) $-2a + b - 1$
8. (a) $a + 3$ (b) $x + 3$ (c) $m + 3$, R:1 (d) $x^2 + 2x + 1$
   (e) $2a^2 - 2a + 3$ (f) $2m^2 - m - 3$, R: −1 (g) $r^2 - 3r + 9$, R: −36 (h) $3x^2 - 4$
9. (a) −11 (b) 2 (c) 8 (d) −9 (e) −4 (f) −1 (g) 22 (h) 6 (i) 2
10. (a) $\frac{9}{4}$ (b) −6 (c) 7 (d) 8 (e) −1 (f) −8 (g) 5 (h) 5 (i) 7
    (j) −7
11. (a) −1 (b) −2 (c) −2 (d) −3 (e) 2 (f) −1 (g) 3
12. (a)

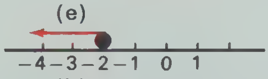
(b)

(c)

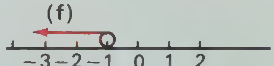
(d)



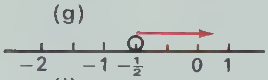
(e)



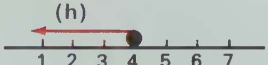
(f)



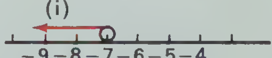
(g)



(h)



(i)



(j)

13. (a) 15 (b) 10 (c) 10 (d) 5 (e) 10 (f) 200

REVIEW AND PREVIEW TO CHAPTER 3

Exercise 1

1. (a) $\frac{2}{5} = \frac{8}{20} = \frac{10}{25}$ (b) $\frac{5}{8} = \frac{10}{16} = \frac{40}{64}$ (c) $\frac{7}{8} = \frac{21}{24} = \frac{35}{40}$ (d) $\frac{3}{7} = \frac{9}{21} = \frac{24}{56}$
   (e) $\frac{1}{6} = \frac{4}{24} = \frac{5}{30}$ (f) $\frac{4}{2} = 2 = \frac{14}{7}$ (g) $3\frac{1}{2} = \frac{7}{2} = \frac{28}{8}$ (h) $2\frac{2}{3} = \frac{8}{3} = \frac{24}{9}$
   (i) $\frac{36}{20} = \frac{9}{5} = \frac{45}{25}$
2. (a) $<$ (b) $<$ (c) $>$ (d) $>$ (e) $<$ (f) $>$ (g) $>$ (h) $>$

Exercise 2

3. (a) $\frac{3}{8}$ (b) $\frac{35}{48}$ (c) $\frac{3}{14}$ (d) $1\frac{1}{8}$ (e) $23\frac{2}{5}$ (f) $3\frac{3}{32}$
   (g) $12\frac{2}{15}$ (h) $7\frac{3}{20}$ (i) $6\frac{1}{4}$ (j) $18\frac{2}{3}$ (k) $6\frac{5}{12}$
4. (a) $4\frac{1}{2}$ (b) $1\frac{3}{17}$ (c) $1\frac{7}{9}$ (d) $1\frac{9}{16}$ (e) $1\frac{2}{3}$ (f) $1\frac{9}{13}$
   (g) $1\frac{5}{19}$ (h) $\frac{23}{30}$ (i) $2\frac{16}{19}$ (j) $1\frac{11}{115}$ (k) $1\frac{11}{12}$

Exercise 3

5. (a) $1\frac{5}{12}$ (b) $4\frac{1}{10}$ (c) $7\frac{11}{12}$ (d) $6\frac{1}{24}$ (e) $10\frac{3}{4}$ (f) $12\frac{13}{24}$
   (g) $18\frac{7}{12}$ (h) $14\frac{1}{8}$ (i) $33\frac{1}{2}$
6. (a) $1\frac{5}{8}$ (b) $1\frac{11}{20}$ (c) $\frac{7}{8}$ (d) $3\frac{5}{8}$ (e) $4\frac{3}{4}$ (f) $12\frac{13}{40}$
   (g) $5\frac{3}{4}$ (h) $7\frac{1}{8}$ (i) $3\frac{9}{20}$
7. (a) $2\frac{1}{10}$ (b) $4\frac{1}{6}$ (c) $1\frac{3}{20}$ (d) $\frac{13}{48}$ (e) $\frac{5}{6}$ (f) $\frac{5}{48}$
   (g) $\frac{23}{24}$ (h) $4\frac{20}{23}$ (i) $1\frac{2}{7}$

Display 3

1. 1.042237297
2. 1.29244403
3. 1.111310133
4. 0.2222222222
5. 1.94713498
6. −2.726183685
7. 6.044044665
8. −3.853315472
9. 22.25136612
10. 0.4197460408

CHAPTER 3

Exercise 3-1

1. (a) $-\frac{3}{4}$ (b) $-\frac{1}{2}$ (c) $\frac{5}{3}$ (d) not defined (e) 0 (f) $\frac{6}{5}$
   (g) $-\frac{1}{5}$ (h) $-\frac{7}{8}$ (i) $-\frac{3}{7}$ (j) $-\frac{4}{9}$ (k) -1 (l) $-\frac{7}{4}$
2. (a) NO (b) YES (c) (i) 0 (ii) not defined
   (d) (i) negative (ii) positive
3. (a) $-\frac{4}{9}$ (b) $-\frac{4}{9}$ (c) same
4. (a) Down to left (b) Down to right (c) to left
   (d) up to right (e) down to right (f) below
5. 0 6. 5 7. -3 8. (b) 4 9. 50

Exercise 3-2

1. (a) $\frac{5}{2}$ (b) $-\frac{1}{6}$ (c) $\frac{3}{4}$ (d) -1 (e) $-\frac{4}{7}$ (f) $-\frac{5}{7}$
2. (a) $y - 4 = \frac{5}{2}(x - 5)$ (b) $y - 3 = -\frac{1}{6}(x - 1)$ (c) $y + 3 = \frac{3}{4}x$
   (d) $y + 2 = -1(x - 2)$ (e) $y - 3 = -\frac{4}{7}(x + 4)$ (f) $y + 6 = -\frac{5}{7}(x - 2)$
3. (a) $y - 2 = 3(x + 5)$ (b) $y + 2 = -1(x - 2)$ (c) $y - 3 = \frac{1}{2}x$
   (d) $y - 2 = \frac{2}{3}(x + 8)$ (e) $y - 1 = 0$ (f) $y = -\frac{4}{3}x$
4. (a) $5x - 3y - 26 = 0$ (b) $x - y + 9 = 0$ (c) $3x - 2y - 1 = 0$
   (d) $3x + 2y - 4 = 0$
5. (i) $y = x$ (ii) $y = -5x$ (iii) $y = mx$
6. (a) 0 (c) Parallel to x-axis (d) $y = 6$ (e) $y = 5$ 7. $y = y_1$
8. (b) Zero (c) $y = 0$ 9. (a) not defined (c) Parallel to y-axis (d) $x = 7$
10. $x = x_1$ 11. (b) not defined (c) $x = 0$
12. (a) $x - 2y + 5 = 0$ (b) $3x - 4y + 7 = 0$ (c) $4x - 3y = 0$ (d) $y = 4$ (e) $x = 3$
13. (a) $y = 2x + 1$ (b) $y = 7x - 4$ (c) $y = -3x + b$ (d) $y = mx + b$
14. (a) $y = 5(x - 2)$ (b) $y = \frac{1}{2}(x + 3)$ (c) $y = m(x - 11)$ (d) $y = m(x - a)$

15. (a) $\frac{x}{4} + \frac{y}{3} = 1$ (b) $\frac{x}{a} + \frac{y}{b} = 1$
16. (a) $y = \frac{5}{3}x - \frac{26}{3}$, $y = x + 9$, $y = \frac{3}{2}x - \frac{1}{2}$, $y = -\frac{3}{2}x + 2$
(c) m represents the slope of the line.
(b) b represents the distance from the origin to where the line cuts the y-axis.

Exercise 3-3

1. (a) $3x - y - 1 = 0$ (b) $4x - y + 14 = 0$ (c) $x - y = 0$ (d) $7x + y - 4 = 0$
2. (a) $3x - 2y - 1 = 0$ (b) $3x + y + 8 = 0$ (c) $2x + 7y = 0$ (d) $4x + 5y - 20 = 0$
3. (a) $y = 3x + 8$ (b) $y = \frac{3}{4}x - \frac{2}{3}$ (c) $y = -\frac{4}{5}x - \frac{7}{3}$ (d) $y = 5$
4. (a) The x-coordinate of the point of intersection of a graph and the x-axis.
(b) (2, 0) (c) $y = 3(x - 2)$
5. (a) (i) $-2, 5$ (ii) $-\frac{2}{3}, \frac{4}{3}$ (iii) $-\frac{3}{2}, -3$ (iv) $\frac{2}{5}, -2$
6. (a) $y = \frac{3}{4}x - 1$ (b) $y = 5(x - 2)$ (c) $4x + 3y = 12$ (d) $8x + 3y + 24 = 0$
7. $x - 2y + 6 = 0$ 8. (a) $y = \frac{1}{5}x + b$ (b) $y = mx + 17$ (c) $y = b$ (d) $y = mx$
9. $x - 11y + 101 = 0$ 10. (a), (b) $5x - 7y + 12 = 0$ (c) Collinear
11. (a) (5, 0), (2, 0), (4, 0), (1, 0), $(\frac{9}{2}, 0)$, $(-5, 0)$ (b) 5, 2, 4, 1, $\frac{9}{2}$, -5
12. (0, 2), 2;) (0, −3), −3); (0, 3), 3; (0, 1), 1; $(0, \frac{9}{4})$, $\frac{9}{4}$; (0, −3), −3
13. (a) $\frac{3}{2}, -2, 3$ (b) $-\frac{5}{3}, 2, \frac{10}{3}$ (c) $\frac{2}{3}, -1, \frac{2}{3}$ (d) not defined, 4, not defined
(e) 0, not defined, 3 (f) $-\frac{2}{3}, -\frac{7}{2}, -\frac{7}{3}$

Exercise 3-4

1. (a) $2, -\frac{1}{2}$ (b) $4, -\frac{1}{4}$ (c) $-3, \frac{1}{3}$ (d) $-1, 1$ (e) $\frac{1}{3}, -3$ (f) $\frac{4}{3}, -\frac{3}{4}$
(g) $-6, \frac{1}{6}$ (h) $5, -\frac{1}{5}$ (i) $-\frac{3}{2}, \frac{2}{3}$ (j) $-\frac{1}{2}, 2$ (k) $\frac{3}{2}, -\frac{2}{3}$ (l) $\frac{1}{3}, -3$
2. $4x - y - 14 = 0$ 3. $3x + 4y - 65 = 0$ 4. $2x + 3y - 12 = 0$ 5. $3x + 5y + 21 = 0$
6. (a) $y = -7$ (b) $x = 2$ 7. $4x - 3y = 0$ 8. $y = -5x + 3$ 9. $5x - 4y - 20 = 0$
10. (b) $3x + 4y + 6 = 0$ (c) $x - y - 5 = 0$ 12. (b) rectangle

Exercise 3-5

1. (a) $V = \frac{3}{4}I$ (b) $\frac{3}{4}$ (d) $V = 6, I = 8$; $V = 3, I = 4$; $V = \frac{3}{2}, I = 2$; $V = 4.5, I = 6$
2. (a) $d = 50t$ (c) 125 km, 10 h 3. (a) \$14 (b) $\frac{1}{2}$ a 4. 70 m
5. 687.5 cm³ 6. 62.5Ω 7. Approx. 24 m
8. (a) $C = 5 + 1n$ (c) \$33 9. (b) 16 (c) \$28 (d) 1 to 15
(e) 17 and over 10. \$1300 11. (a) $RI = 10$ (b) $I = 10, 5, \frac{5}{2}, 2, \frac{5}{3}, \frac{5}{4}, 1,$
respectively (f) Straight line 12. (c) $6\frac{2}{3}$ h 13. 262.5 kPa

Review Exercise

1. (a) $-\frac{4}{3}$ (b) $-\frac{1}{2}$ (c) $-\frac{7}{3}$ (d) 3 (e) $\frac{9}{2}$ (f) $\frac{1}{2}$ 2. (a) 5 (b) 0 3. (b) no
4. (a) $x - 2y + 10 = 0$ (b) $2x + y - 5 = 0$ (c) $3x - 4y - 6 = 0$ (d) $2x + 3y + 11 = 0$
(e) $2x - 5y - 29 = 0$ 5. $y = -x + 4$ 6. $x + 3y - 9 = 0$ 7. $x + 4y - 9 = 0$
8. $5x + 3y + 19 = 0$ 9. $2x + 3y - 21 = 0$ 10. $3x - 5y - 15 = 0$ 11. $-\frac{3}{2}, \frac{7}{2}$
12. (a) $\frac{4}{3}$ (b) -3 13. 1 14. $6x - 7y - 23 = 0$ 15. $5x - 3y + 8 = 0$
16. (0, 2) 17. $7x - 3y + 24 = 0$ 18. $2x - 3y + 12 = 0$ 19. $(\frac{7}{3}, 0)$
20. $9x + 2y + 8 = 0$ 21. -4

REVIEW AND PREVIEW TO CHAPTER 4

1. (a) 5 (b) 75 (c) 48 (d) 1 (e) 84 (f) 4.41
(g) 1.3 (h) 1.65 (i) 21 (j) 8.75 (k) 0.875 (l) 0.05
(m) 1.35 (n) 1.65 (o) 2.475 (p) 71 (q) 240 (r) 30
2. (a) 50% (b) 25% (c) 25% (d) 10% (e) 20% (f) $1\frac{2}{3}$%
(g) 300% (h) 500%
3. (a) 5 cm (b) 13 cm (c) 7.2 cm (d) 9.5 cm (e) 9.7 cm (f) 12 cm
4. (a) 12 cm² (b) 6 cm² (c) 30 cm² (d) 300 cm² (e) not possible
(f) 72 m² (g) not possible (h) 30 m²

Display 4

1. 2.078740157
2. -0.4647256439
3. 1.3477748
4. -2.624466572
5. -0.2443338861
6. 0.156102176
7. 4.208633094
8. 0.4085561497
9. 1.384743412
10. -0.9711815562

CHAPTER 4

Exercise 4-1

1. (a) 2 (b) 6 (c) 4 (d) 5 (e) 4 (f) 10
   (g) 4 (h) 6 (i) 8 (j) 8
2. (a) $\sqrt{5}$ (b) $\sqrt{29}$ (c) $\sqrt{2}$ (d) $\sqrt{53}$ (e) 6 (f) 7
   (g) $2\sqrt{13}$ (h) $\sqrt{17}$ (i) 5 (j) $\sqrt{13}$ (k) $\sqrt{41}$ (i) $10\sqrt{2}$
3. $5\sqrt{2}, 2\sqrt{17}, \sqrt{106}$ 4. $22 + 2\sqrt{85} \simeq 40.4$
5. $2\sqrt{10}$ 6. (a) isosceles (b) scalene (c) scalene (d) equilateral
7. 13, 17 9. $(-2,0)$ 10. $(0, -5)$

Exercise 4-2

1. (a) (6, 7) (b) (5, 5) (c) $(-1, 4)$ (d) $(\frac{11}{2}, \frac{1}{2})$ (e) $(2, -\frac{1}{2})$ (f) $(-\frac{7}{2}, -7)$
   (g) $(-\frac{1}{2}, -2)$ (h) $(-2, -\frac{3}{2})$ (i) $(-\frac{1}{2}, \frac{1}{2})$ (j) $(\frac{1}{2}, -\frac{11}{10})$
2. (a) $M(1, 3)$, $Q(-1, 1)$ (b) Slope QM = Slope BC
3. (a) Slope $XY = \frac{1}{2}$, slope $XZ = -2$ (b) $XY \perp XZ$ (c) Right-angled (d) $W(-2, -4)$
   (e) $XW = YW = ZW$ 4. (1, 3) 5. $(-4, -5)$ 6. (5, 3) 7. $(-4, 4)$

Exercise 4-3

1. (a) yes (b) no (c) yes (d) no (e) yes (f) no
2. (a) -4 (b) 2 5. (a) $D(7, 2)$, $E(6, 4)$, $F(4, 1)$, (b) $x + 4y - 15 = 0$,
   $5x - y - 26 = 0$, $4x - 5y - 11 = 0$ (d) $G(\frac{17}{3}, \frac{7}{3})$ (e) $-\frac{2}{3}, -3, \frac{1}{2}$
   (f) $2x + 3y - 20 = 0$, $3x + y - 22 = 0$, $x - 2y - 2 = 0$ (g) $R(\frac{46}{7}, \frac{16}{7})$
   (h) $RA = RB = RC$ (i) $2x + 3y - 15 = 0$, $3x + y - 14 = 0$, $x - 2y + 1 = 0$
   (j) $O(\frac{27}{7}, \frac{17}{7})$ (l) $GO = 2GR$

Exercise 4-4

1. (a) 18 sq. units (b) 11 sq. units (c) $8\frac{1}{2}$ sq. units (d) $13\frac{1}{2}$ sq. units
2. (a) 6 sq. units (b) $8\frac{1}{2}$ sq. units (c) $9\frac{1}{2}$ sq. units (d) $17\frac{1}{2}$ sq. units
   (e) 4 sq. units (f) 26 sq. units (g) 7 sq. units (h) $7\frac{1}{2}$ sq. units
4. (a) 24 sq. units (b) 13 sq. units

Review Exercise

1. (a) $\sqrt{17}$ (b) $\sqrt{53}$ (c) $2\sqrt{13}$ (d) 4 (e) $\sqrt{85}$ (f) $4\sqrt{2}$ (g) $3\sqrt{2}$
   (h) 6 2. (a) (4, 7) (b) $(-6, 2)$ (c) $(3, -1)$ (d) $(\frac{7}{2}, -\frac{3}{2})$ (e) $(\frac{1}{2}, \frac{9}{2})$
   (f) $(-5, -8)$ 3. (a) yes (b) no (c) yes (d) no (e) no
4. (a) 14 sq. units (b) 32 sq. units (c) 19 sq. units 5. (a) 0 sq. units
   (b) collinear 6. Isosceles 7. $(9, -12)$ 8. (a) $(3, \frac{7}{2})$, $(4, 1)$, $(5, \frac{7}{2})$
   (b) 10 sq. units (c) $\frac{5}{2}$ sq. units (d) $\triangle ABC = 4 \times$ the smaller area
9. 45 sq. units 10. (a) $XY = ZW$ (b) slope XY = slope ZW
   (c) $XYZW$ is a parallelogram 13. 10.20 14. 26 sq. units

REVIEW AND PREVIEW TO CHAPTER 5

Exercise 1

1. (a) 98.0 (b) 55.7 (c) 476 (d) 17.9 (e) 0.156 (f) 1330
   (g) 1600 (h) 0.0366
2. (a) 1470 (b) 30.7 (c) 226 (d) 367 (e) 2.31 (f) 1670
3. (a) 2.08 (b) 47.0 (c) 3.89 (d) 0.253 (e) 0.130 (f) 1.32
   (g) 0.330 (h) 0.0551

4. (a) 72.1 (b) 10.7 (c) 0.599 (d) 0.239 (e) 8.69 (f) 352
5. (a) 692 (b) 94.1 (c) 132 (d) 16 100 (e) 0.270 (f) 0.00533
(g) 21.6 (h) 61.8
6. (a) 4.16 (b) 3.14 (c) 11.1 (d) 0.748 (e) 27.6 (f) 0.286
(g) 0.0412 (h) 0.0238

Exercise 2

1. (a) $\{2, 4, 6, 8\}$ (b) $\{11, 13, 17, 19, 23, 29\}$ (c) $\{3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48\}$ (d) $\{4, 9, 16, 25, 36, 49, 64, 81, 100\}$ (e) $\{6\}$
2. (a) $\{5, 6\}$ (b) $\{c\}$ (c) $\{3, 7, 9, 10, 12\}$ (d) $\{3, 4, 5, 6, 7, 8\}$ (e) $\{\ \}$ or ϕ
(f) $\{a\}$

Exercise 3

1. (a) -3 (b) 8 (c) 15 (d) 20 (e) $\frac{13}{2}$ (f) 14 (g) $\frac{3}{2}$ (h) $\frac{25}{7}$
2. (a) $x \geqq 5$ (b) $x > -5$ (c) $x < 2$ (d) $x \geqq -\frac{1}{3}$ (e) $m < 7$ (f) $x \geqq -3$

Display 5

1. 32.73290852
2. 21.16246677
3. 9.976729424
4. 3.764306
5. 0.211697898
6. 29.806207
7. 0.0115585
8. 158.7419604
9. 17.1656634
10. 172.0476986

CHAPTER 5

Exercise 5-1

1. (a) $y = -2x + 7$ (b) $y = -4x - 7$ (c) $y = 3x - 4$ (d) $y = 2x - 3$
(e) $y = -\frac{3}{2}x + 2$ (f) $y = -\frac{4}{3}x - \frac{5}{3}$ (g) $y = x - 6$ (h) $y = \frac{5}{3}x - \frac{5}{3}$
(i) $y = \frac{2}{3}x$ (j) $y = \frac{2}{5}x - \frac{1}{5}$

Exercise 5-2

1. (a) $(-3, 2)$ (b) $(4, 2)$ (c) $(1\frac{1}{2}, 2\frac{1}{2})$ (d) $(-\frac{12}{7}, \frac{22}{7})$
2. (a) $0 \leqq n < 6$ (b) $n = 6$ (c) $n > 6$

Exercise 5-3

1. (a) $y = -2x + 7$ (b) $y = 2x + 4$ (c) $y = -3x - 7$ (d) $y = -\frac{2}{3}x + 2$
(e) $y = 2x - \frac{7}{2}$ (f) $y = -\frac{1}{3}x + \frac{7}{3}$ (g) $y = \frac{5}{3}x - \frac{4}{3}$ (h) $y = -\frac{2}{3}x - \frac{4}{3}$
(i) $y = -\frac{3}{2}x - \frac{7}{2}$
2. (a) (3, 8) (b) (4, 11) (c) (2, 5) (d) $(-1, 5)$
3. (a) $(5, -17)$ (b) $(-2, -5)$ (c) $(\frac{3}{4}, \frac{9}{2})$ (d) $(\frac{1}{2}, \frac{7}{2})$ (e) $(-1, 3)$
(f) (3, 0) 4. (a) (2, 4) (b) (12, 4) (c) (7, 3) (d) $(-1, -1)$
(e) $(1, -\frac{1}{2})$ (f) (16, 11) 5. (a) (1, 1) (b) (2, 3) (c) $(2, -1)$
(d) $(1, -2)$ (e) $(-\frac{1}{2}, 1)$ (f) $(-\frac{1}{2}, -\frac{1}{3})$ (g) $(-\frac{2}{3}, \frac{1}{2})$ (h) $(\frac{2}{5}, -\frac{3}{4})$
6. $(3, 4), (-1, 2), (1, -2)$

Exercise 5-4

1. (a) (3, 2) (b) (3, 2) (c) (2, 2) 2. (a) (3, 0) (b) (2, 1)
(c) $(5, -1)$ 3. (a) (3, 2) (b) $(\frac{3}{2}, 3)$ (c) (3, 5) 4. (a) $(-\frac{6}{7}, \frac{30}{7})$
(b) $(\frac{3}{4}, \frac{2}{3})$ (c) $(-3, -3)$ 5. (a) (8, 3) (b) (9, 4) (c) (20, 12)
6. (a) (3, 4) (b) $(3, -1)$

Exercise 5-5

1. (a) $x + 5$ (b) $x + 4$ (c) $y - 3$ (d) $6y$ (e) $3b$
   (f) $m + 4$ (g) $x + 2$ (h) $q - 5$ (i) $r - 7$ or $7 - r$ (j) $\frac{b}{4}$
   (k) $8x$ (l) $2d$ (m) $x + 3x$ (n) $\frac{2x}{y}$ (o) $4b + 6$
   (p) $5l + 3$ (q) $4q - 10$ (r) $x + 2$ (s) $x - 3$ (t) $2s - 40$
   (u) $6x - 4$ (v) $\frac{1}{2}r + 2$ (w) $x + 2x$ (x) $2(x + 2)$ (y) $\frac{5}{100}x$
2. (a) $2x = 36$ (b) $3x = 27$ (c) $x - 2 = 8$ (d) $x + 3 = 12$ (e) $\frac{x}{5} = 20$
   (f) $2x + 2 = 12$ (g) $5x - 4 = 16$ (h) $x - 3 = 7$ (i) $2x + 4 = 84$
   (j) $\frac{1}{2}x - 10 = 60$
3. (a) $x + y = 21$ (b) $x + y = 40$ (c) $x - y = 1$
   (d) $2x + 3y = 70$ (e) $\frac{1}{2}x + 3 = 4y$ (f) $2x - \frac{2}{3}y = 5$ (g) $x - \frac{1}{2}y = 12$
   or $\frac{1}{2}x - y = 12$ (h) $x + y = 15$ (i) $3x + 5y = 400$

Exercise 5-6

1. 3, 5 **2.** 2, 7 **3.** 7, 2 **4.** 7, 4 **5.** 5, 8 **6.** 7 **7.** 41, 15 **8.** 15 **9.** 13, 15 **10.** 17 **11.** 24, 14 **12.** 80 kg, at 40¢, 20 kg at 80¢ **13.** 45 kg at 75¢, 55 kg at 95¢ **14.** $1 600, $400 **15.** $1700, $1800 **16.** $3000, $2000 **17.** $150, $50 **18.** 300 km at 60 km/h, 200 km at 50 km/h **19.** 3 h at 80 km/h, $\frac{1}{2}$ h at 40 km/h **20.** 2 h, 36 km **21.** 6.5 km/h, 2.5 km/h **22.** 260 km/h, 40 km/h

Exercise 5-7

Solutions not provided for this exercise.

Exercise 5-8

Solutions not provided for this exercise.

Exercise 5-9

Solutions not provided for this exercise.

Exercise 5-10

1. (b) $x + 3y$ is a maximum at (0, 5). $x + y$ is a maximum at (4, 3). $3x + y$ is a maximum at (6, 0). **2.** 2 model 1, 5 model 1A **3.** John 3 h, Bill 2 h **4.** Brand X 180, Brand Y 60

Review Exercise

2. (a) (1, 1) (b) (1, −2) (c) (−2, −2) (d) (2, −1) (e) (1, 0)
   (f) (2, −3) (g) (10, 6) (h) (14, 15) (i) (−1, 2)

3. 7, 11 **4.** 9, 7 **5.** 13 **6.** $1000, $3000 **7.** $600, $500 **8.** 220 km at 55 km/h, 135 km at 45 km/h

REVIEW AND PREVIEW TO CHAPTER 6

Exercise

1. (a) 13 (b) 45.0 (c) 58.0 (d) 91.0 (e) 27.0
   (f) 35.0 (g) 62.0 (h) 42.0 (i) 19.0 (j) 24.0
2. (a) 18.2 (b) 27.6 (c) 4060 (d) 0.000156 (e) 9.49
   (f) 7310 (g) 9.86 (h) 40.2 (i) 650 (j) 279
3. (a) 14.1 (b) 5.35 (c) 7.44 (d) 27.6 (e) 4530
   (f) 0.226 (g) 1.91 (h) 274 (i) 0.0671 (j) 28.8
   (k) 8.16 (l) 181 (m) 0.0462 (n) 0.001 37 (o) 5.36
4. (a) 4.06 (b) 2.46 (c) 22.8 (d) 62.5

Display 6

1. 40.627876
2. 0.5439379867
3. 43989706.05
4. 0.0005433384
5. 154.3351321
6. 0.1889137169
7. −58.94650137
8. 5.670117378
9. 80498674.14
10. 2.119574107

CHAPTER 6

Exercise 6-1

1. (a) 4 (b) 7 (c) 9 (d) 10 (e) 11
 (f) $\frac{3}{4}$ (g) $\frac{5}{6}$ (h) $\frac{9}{10}$ (i) $\frac{8}{9}$ (j) $\frac{1}{5}$
 (k) 15 (l) 17 (m) 21 (n) 0.1 (o) 0.05

2. (a) $\sqrt{6}$ (b) $\sqrt{55}$ (c) $\sqrt{14}$ (d) $\sqrt{70}$ (e) $\sqrt{35}$ (f) $\sqrt{33}$
 (g) $\sqrt{30}$ (h) $\sqrt{10}$ (i) $\sqrt{42}$ (j) $\sqrt{65}$ (k) $\sqrt{143}$ (l) 7

3. (a) $6\sqrt{15}$ (b) $3\sqrt{14}$ (c) $8\sqrt{15}$ (d) $2\sqrt{21}$ (e) $3\sqrt{22}$ (f) $6\sqrt{15}$
 (g) $6\sqrt{6}$ (h) $8\sqrt{35}$ (i) $12\sqrt{66}$ (j) $35\sqrt{6}$ (k) $12\sqrt{30}$ (l) 30

4. (a) $2\sqrt{3}$ (b) $3\sqrt{3}$ (c) $5\sqrt{3}$ (d) $3\sqrt{5}$ (e) $7\sqrt{2}$ (f) $4\sqrt{2}$
 (g) $2\sqrt{17}$ (h) $2\sqrt{5}$ (i) $10\sqrt{2}$ (j) $2\sqrt{7}$ (k) 21 (l) 32
 (m) $6\sqrt{2}$ (n) $5\sqrt{2}$ (o) $2\sqrt{2}$ (p) $2\sqrt{6}$

5. (a) $\sqrt{20}$ (b) $\sqrt{63}$ (c) $\sqrt{18}$ (d) $\sqrt{50}$ (e) $\sqrt{99}$ (f) $\sqrt{500}$
 (g) $\sqrt{300}$ (h) $\sqrt{98}$ (i) $\sqrt{200}$ (j) $\sqrt{56}$ (k) $\sqrt{294}$ (l) $\sqrt{396}$
 (m) $\sqrt{1200}$ (n) $\sqrt{1250}$ (o) $\sqrt{160}$ (p) $\sqrt{162}$

6. (a) $7\sqrt{2}$ (b) $2\sqrt{15}$ (c) $7\sqrt{15}$ (d) 7 (e) $3\sqrt{2}$ (f) $5\sqrt{3}$
 (g) $25\sqrt{6}$ (h) $5\sqrt{10}$ (i) $6\sqrt{3}$ (j) $70\sqrt{2}$ (k) $30\sqrt{2}$ (l) $30\sqrt{5}$
 (m) 6 (n) $5\sqrt{6}$ (o) 30 (p) 36 (q) $90\sqrt{3}$ (r) 144

7. (a) $5x$ (b) $7x^2$ (c) x^3 (d) $3x$ (e) $10x^2$ (f) $3x^2\sqrt{2}$
 (g) $5x^2\sqrt{3x}$ (h) $6x^2$ (i) $3x\sqrt{5x}$ (j) $3x^2\sqrt{2}$ (k) $2\sqrt{5x}$ (l) $3x^2\sqrt{3}$
 (m) $5x^2\sqrt{5x}$ (n) $x\sqrt{15}$ (o) $3x\sqrt{3x}$ (p) $2x^3\sqrt{2}$ (q) $5x^2\sqrt{5}$ (r) $5x^2\sqrt{5}$
 (s) $8x\sqrt{x}$ (t) $3x\sqrt{11x}$

Exercise 6-2

1. (a) $8\sqrt{7}$ (b) $4\sqrt{3}$ (c) $\sqrt{13}$ (d) $5\sqrt{5}$ (e) $14\sqrt{11}$ (f) $5\sqrt{3}$
 (g) $4\sqrt{7}$ (h) $\sqrt{15}$ (i) $6\sqrt{5}$ (j) $12\sqrt{3}$ (k) $2\sqrt{5}$ (l) $2\sqrt{3}$
 (m) $12\sqrt{3}$ (n) $3\sqrt{7}$

2. (a) $4\sqrt{3} - 2\sqrt{11}$ (b) $11\sqrt{17} - 3\sqrt{15}$ (c) $8\sqrt{2} + 4\sqrt{3}$
 (d) $10\sqrt{5} + 3\sqrt{7}$ (e) $10\sqrt{3} + 6\sqrt{5}$ (f) $7\sqrt{7} - 16\sqrt{3}$

3. (a) $5\sqrt{2} + 8$ (b) $8\sqrt{3} + 4$ (c) $16 - 8\sqrt{2}$
 (d) $4\sqrt{3} + 4\sqrt{5}$ (e) $3\sqrt{7}$ (f) $-5 - \sqrt{2}$

4. (a) $5\sqrt{2}$ (b) $5\sqrt{3}$ (c) $8\sqrt{2}$ (d) $3\sqrt{2}$
 (e) $5\sqrt{5}$ (f) $\sqrt{3}$ (g) $11\sqrt{2}$ (h) $13\sqrt{3}$
 (i) $2\sqrt{6} - 3\sqrt{2}$ (j) $\sqrt{2}$ (k) $23\sqrt{3}$ (l) $-\sqrt{2}$
 (m) $11\sqrt{2}$ (n) $4\sqrt{2}$ (o) $15\sqrt{2}$ (p) $18\sqrt{2} - 2\sqrt{3}$

5. (a) $\sqrt{x}$ (b) $4\sqrt{a}$ (c) $21\sqrt{x}$ (d) $6\sqrt{a}$ (e) $10\sqrt{2x}$ (f) $\sqrt{3a}$

Exercise 6-3

1. (a) $\sqrt{6} + \sqrt{10}$ (b) $5\sqrt{3} + \sqrt{21}$ (c) $6\sqrt{5} - 9\sqrt{2}$ (d) $5\sqrt{3} - 10\sqrt{5}$
 (e) $3 + \sqrt{3}$ (f) $4 - 2\sqrt{3}$ (g) $2\sqrt{3} - 2\sqrt{6}$ (h) $3\sqrt{10} + 3\sqrt{30}$
 (i) $2\sqrt{3} + 3\sqrt{6}$ (j) $6\sqrt{2} + 6\sqrt{6}$

2. (a) $17 + 8\sqrt{5}$ (b) $27 + 7\sqrt{21}$ (c) $8 + 3\sqrt{6}$ (d) $16 - 17\sqrt{2}$
(e) $142 + 32\sqrt{21}$ (f) $-16 - 52\sqrt{15}$ (g) $108 + 10\sqrt{15}$ (h) $-163 - 9\sqrt{11}$
(i) $36 - 18\sqrt{10}$ (j) $-10\sqrt{10} + 15\sqrt{14} - 8\sqrt{15} + 12\sqrt{21}$

3. (a) $8 + 2\sqrt{15}$ (b) $9 - 2\sqrt{14}$ (c) $8 - 4\sqrt{3}$ (d) $13 - 2\sqrt{42}$
(e) $19 + 6\sqrt{2}$ (f) $3 - 2\sqrt{2}$ (g) $82 - 8\sqrt{10}$ (h) $342 + 240\sqrt{2}$
(i) $372 + 24\sqrt{30}$ (j) $8 + 4\sqrt{3}$ (k) $42 + 24\sqrt{3}$ (l) $116 - 48\sqrt{5}$

4. (a) 3 (b) 1 (c) 2 (d) 8 (e) -5
(f) 78 (g) 146 (h) 148 (i) 94 (j) 31

5. (a) $94 - 17\sqrt{6}$ (b) $-73 - 62\sqrt{3}$ (c) 18

6. (a) $6a - 4\sqrt{a}$ (b) $a - b^2$ (c) $3a + 3a\sqrt{a}$
(d) $a + 4 + 4\sqrt{a}$ (e) $2a + a\sqrt{2}$ (f) $6a - 10 - 11\sqrt{a}$

Exercise 6-4

1. (a) $\sqrt{5}$ (b) 5 (c) $4\sqrt{3}$ (d) 2 (e) $\frac{3}{2}$ (f) 6
(g) 5 (h) 21

2. (a) 2.65 (b) 2.45 (c) 2.65 (d) 33.5 (e) 28.3 (f) 2.31
(g) 2.65 (h) 15.0 (i) 6.71 (j) 1.73 (k) 5.20 (l) 9
(m) 4.45 (n) 1.27 (o) 0.764

3. (a) $\dfrac{3\sqrt{5}}{5}$, 1.34 (b) $\dfrac{8\sqrt{7}}{7}$, 3.02 (c) $\dfrac{\sqrt{21}}{3}$, 1.53 (d) $\dfrac{2\sqrt{55}}{11}$, 1.35 (e) $\dfrac{\sqrt{15}}{9}$, 0.43 (f) $\dfrac{3\sqrt{3}}{4}$, 1.30
(g) $\dfrac{5\sqrt{14}}{14}$, 1.34 (h) $\dfrac{3\sqrt{30}}{\sqrt{5}}$, 3.29 (i) $\dfrac{\sqrt{15}}{15}$, 0.258 (j) $\dfrac{\sqrt{21}}{3}$, 1.53 (k) $\dfrac{5}{3}$, 1.67 (l) $\dfrac{3\sqrt{105}}{5}$, 6.15
(m) $\dfrac{7\sqrt{22}}{22}$, 1.49 (n) $\dfrac{5\sqrt{2}}{8}$, 0.884 (o) $\dfrac{7\sqrt{14}}{4}$, 6.55

4. (a) $\sqrt{3} + \sqrt{2}$ (b) $\dfrac{\sqrt{15} - \sqrt{6}}{3}$ (c) $\dfrac{\sqrt{30} + \sqrt{10}}{4}$
(d) $5\sqrt{2} + 5$ (e) $\dfrac{14\sqrt{5} - 7\sqrt{2}}{18}$ (f) $\dfrac{3\sqrt{2} + \sqrt{3}}{5}$
(g) $\dfrac{20 + 5\sqrt{3}}{-13}$ (h) $\dfrac{27 + 10\sqrt{2}}{23}$ (i) $\dfrac{60 + 15\sqrt{2} + 25\sqrt{3} + 12\sqrt{6}}{57}$
(j) $3\sqrt{3} + 3\sqrt{2}$ (k) $\sqrt{7} + \sqrt{3}$ (l) $\dfrac{53 + 10\sqrt{6}}{47}$

Exercise 6-5

1. (a) 16 (b) 9 (c) 32 (d) 27 (e) 64 (f) 125
(g) 81 (h) 49 (i) 1000 (j) 81

2. (a) a^7 (b) a^9 (c) a^6 (d) a^8 (e) b^{11} (f) b^{14}
(g) b^{10} (h) 10^5

3. (a) $3a^6$ (b) $10m^7$ (c) $8a^5$ (d) $4x^3$ (e) $10n^5$ (f) $35x^7$
(g) $3x^4$ (h) $6x^5$ (i) $6x^5$

4. (a) $2a^3$ (b) $3a^3$ (c) t (d) $3x^2$ (e) $3m$ (f) $2n$
(g) $4a$ (h) t^2 (i) $3x^4$

5. (a) 64 (b) 81 (c) m^{15} (d) $5n^{14}$ (e) $16p^{12}$ (f) $625a^4$
(g) $27x^9$ (h) $8x^6$ (i) -1 (j) -1 (k) -1 (l) $+1$

6. (a) a^3b^2 (b) $2a^3b$ (c) $2a^4b^4$ (d) $3a^2b$ (e) b (f) $2mn^2$

7. (a) $2^{a+b\quad c}$ (b) 2^{2a+3b} (c) 2^b (d) 3^c (e) 3 (e) a^{2b-1}

Exercise 6-6

1. (a) $\frac{1}{25}$ (b) 1 (c) 1 (d) $\frac{1}{81}$ (e) $\frac{1}{8}$ (f) $\frac{1}{9}$
   (g) 1 (h) $\frac{1}{1000}$ (i) 1 (j) 10
2. (a) a^{-2} (b) a^6 (c) a^4 (d) a^{-4} (e) a^9 (f) x^4
   (g) 1 (h) 1 (i) b^{-1} (j) b^{-3} (k) b^{-5} (l) b
3. (a) $15a^3$ (b) x^5y^{-3} (c) 6 (d) $21a$ (e) $6a^{-5}$ (f) $15a^{-2}$
   (g) 10 (h) $6b^{-1}$ (i) $12a^{-1}b^5$
4. (a) a^8 (b) m^{-5} (c) r^{-3} (d) $9a^7$ (e) $\dfrac{1}{3p^3}$ (f) $\dfrac{1}{2t^2}$
   (g) $\dfrac{x^5}{3}$ (h) x^{-5} (i) $\dfrac{1}{8a^9}$
5. (a) $\frac{3}{2}$ (b) $\frac{5}{6}$ (c) $\frac{1}{9}$ (d) $\frac{1}{4}$ (e) 3 (f) 2
   (g) 1 (h) $\frac{1}{2}$ (i) $\frac{4}{3}$ (j) 10 (k) 100 (l) $\frac{225}{8}$

Exercise 6-7

1. (a) 3 (b) 2 (c) 4 (d) 5 (e) 9 (f) 3 (g) 25 (h) 5
2. (a) $\frac{1}{3}$ (b) $\frac{1}{25}$ (c) $\frac{1}{4}$ (d) $\frac{1}{2}$ (e) $\frac{1}{2}$ (f) $\frac{1}{6}$
   (g) 9 (h) 4 (i) 27 (j) 8 (k) 125 (l) $\frac{1}{4}$
3. (a) 3 (b) $\frac{1}{32}$ (c) 8 (d) $\frac{1}{8}$ (e) 16 (f) 625
   (g) 8 (h) 2 (i) $\frac{1}{16}$ (j) 4 (k) $\frac{1}{81}$ (l) 3
4. (a) 2 (b) $10^{0.75}$ (c) $10^{0.25}$ (d) $10^{\frac{2}{3}}$ (e) 5 (f) 2
   (g) 9 (h) 9 (i) $5^{\frac{7}{6}}$
5. (a) a^4 (b) $3ab^2$ (c) $25a^2$ (d) $4a^2$ (e) $27a^3$ (f) $2ab^2$

Review Exercise

1. (a) 12 (b) 17 (c) -13 (d) 15 (e) -20 (f) 21 (g) 25 (h) 24
2. (a) $\sqrt{110}$ (b) $7\sqrt{30}$ (c) $\sqrt{7}$ (d) $\sqrt{21}$ (e) $\frac{1}{4}\sqrt{\frac{15}{2}}$ (f) $\frac{1}{2}$
3. (a) $2\sqrt{6}$ (b) $3\sqrt{6}$ (c) $2\sqrt{13}$ (d) $7\sqrt{2}$ (e) $5\sqrt{5}$
4. (a) $\sqrt{45}$ (b) $\sqrt{28}$ (c) $\sqrt{108}$ (d) $\sqrt{160}$ (e) $\sqrt{27}$
5. (a) $-2\sqrt{15}$ (b) $\sqrt{5}$ (c) $16\sqrt{3}$ (d) $\sqrt{5}$
   (e) $5\sqrt{2}$ (f) $7\sqrt{7} - 6\sqrt{2}$ (g) $7\sqrt{2}$ (h) $12\sqrt{3} - 2\sqrt{2}$
6. (a) 42 (b) 23 (c) $47 + 12\sqrt{15}$ (d) $75 - 12\sqrt{21}$
   (e) $5\sqrt{15} - 1$ (f) $54 - 14\sqrt{21}$ (g) $25 + 4\sqrt{15}$ (h) $37 + 3\sqrt{10} + 6\sqrt{2}$
7. (a) $\frac{3}{2}$ (b) 6 (c) $\dfrac{\sqrt{14}}{2}$ (d) $5\sqrt{2}$
   (e) $\sqrt{3a}$ (f) $2\sqrt{3}$ (g) $\dfrac{\sqrt{30}}{20}$ (h) $\dfrac{\sqrt{6}}{2}$
   (i) 21 (j) $\dfrac{\sqrt{11} - \sqrt{3}}{2}$ (k) $\dfrac{3(5 - \sqrt{3})}{22}$ (l) $\dfrac{20\sqrt{3} - 6\sqrt{2}}{47}$
8. (a) -1 (b) 3^7 (c) 3^5 (d) S^4 (e) 2 (f) 2^8n^{12}
   (g) 1 (h) 1 (i) $\frac{1}{16}$ (j) $\frac{1}{2}$ (k) 3 (l) $4a^2$
   (m) $\dfrac{1}{7m^3}$ (n) $\frac{1}{2}$ (o) 49
9. (a) 100 (b) 10
   (c) 16 (d) 10 (e) 1 (f) 1 (g) 4×10^8 (h) 2×10^2

REVIEW AND PREVIEW TO CHAPTER 7

Exercise 1

1. 1.12×10^1
2. 3.75×10^0
3. 2.575×10^{-1}
4. 3.25×10^{-2}
5. 1.25×10^{-3}
6. 5.78×10^{-4}
7. 5.63×10^{-3}
8. 4.25×10^{-1}
9. 9.3×10^7
10. 1.86×10^5
11. 3.5127×10^4
12. 4.25×10^5
13. 3.25×10^1
14. 3.125×10^0
15. 4.703×10^2

Exercise 2

1. (a) 10^{12} (b) 10^{3} (c) 10^{-8} (d) 10^{8} (e) 10^{25} (f) 10^{15}
   (g) 10^{3} (h) $10^{6.0}$ (i) $10^{3.0}$
2. (a) 10^{11} (b) 10^{-8} (c) 10^{4} (d) 10^{14} (e) 10^{-8} (f) 10^{4}
   (g) 10^{2} (h) 10^{7} (i) 10^{12} (j) 10^{-6} (k) 10^{-1} (l) 10^{3}
3. (a) $10^{5+.62}$ (b) $10^{5+.50}$ (c) $10^{-2+.75}$ (d) $10^{-5+.81}$
   (e) $10^{-5+.50}$ (f) $10^{8+.08}$ (g) $10^{6+.50}$ (h) $10^{-2+.74}$

Display 7

1. 2.565151068
2. 2.147091055
3. 5.234182744
4. 1.152150071
5. 10.93923551
6. 6.369477011
7. 3.488415278
8. 8.350586686
9. 415.9369577
10. 608.9425931

CHAPTER 7

Exercise 7-1

1. 780 2. 2 816 3. 16*n* 4. 256 5. 8 units 6. 5 h 7. (a) one (b) 400

Exercise 7-2

1. (a) 3.16 (b) 3.98 (c) 6.31 (d) 10.0 (e) 1.78 (f) 2.24
   (g) 5.62 (h) 7.08 (i) 2.00 (j) 1.41 (k) 2.51 (l) 1.59
   (m) 8.91 (n) 4.47 (o) 2.82 (p) 3.55
2. (a) 0.48 (b) 0.78 (c) 0.30 (d) 0.60 (e) 0.90 (f) 0.54
   (g) 0.43 (h) 0.93 (i) 0.86 (j) 0.95 (k) 0.91 (l) 0.67
   (m) 0.72 (n) 0.81 (o) 0.94 (p) 0.57

Exercise 7-3

1. 12.5 cm 2. 8.748 m 3. $\frac{1}{32}$ 4. (a) 2.25 m, 0.7 m (b) $\frac{1}{4}$ (c) 5, 7
   (d) 75% 5. 2.5% 6. 6 cm

Exercise 7-4

1. (a) 1.78 (b) 6.31 (c) 5.62 (d) 1.26 (e) 1.59 (f) 3.98
   (g) 2.51 (h) 2.24 (i) 3.02 (j) 4.37 (k) 4.17 (l) 7.24
2. (a) 0.18 (b) 0.30 (c) 0.48 (d) 0.78 (e) 0.90 (f) 0.88
   (g) 0.58 (h) 0.38 (i) 0.76 (j) 0.96 (k) 0.80 (l) 0.64
3. 19.6 4. 15.5 5. 1.3 6. 22 7. 17.50 8. 290 9. 5.6 10. 1500 11. 1370
12. 35 000 13. 26 000 14. 6.5 15. 16 16. 260 cm^2 17. 390 000 cm^3
18. 48 000 cm^3 19. 46 cm^2 20. 9 300 000 cm^3

Exercise 7-5

1. (a) 2.065 (b) 2.673 (c) 1.738 (d) 2.731 (e) 8.472 (f) 5.188
   (g) 1.766 (h) 4.217 (i) 2.673 (j) 2.153 (k) 5.623 (l) 2.265
   (m) 7.161 (n) 1.334 (o) 3.365 (p) 4.634
2. (a) 0.300 (b) 0.354 (c) 0.387 (d) 0.414 (e) 0.041 (f) 0.044
   (g) 0.147 (h) 0.757 (i) 0.837 (j) 0.995 (k) 0.881 (l) 0.811
3. (a) 542.0 (b) 13.30 (c) 365.6 (d) 91 200 (e) 54.83 (f) 182.4
   (g) 23.44 (h) 8128 (i) 0.002 735 (j) 0.027 29 (k) 0.6683
4. (a) $10^{2.4393}$ (b) $10^{1.9159}$ (c) $10^{0.5551}$ (d) $10^{2.4900}$
   (e) $10^{0.7050}$ (f) 10^{3} (g) $10^{1.9590}$ (h) $10^{1.3692}$
   (i) $10^{0.9956} \times 10^{-1}$ (j) $10^{0.8069} \times 10^{-2}$ (k) $10^{.8579} \times 10^{-4}$ (l) $10^{0.1038} \times 10^{-3}$
5. (a) 102 (b) 923 (c) 23.1 (d) 3.03
   (e) 0.742 (f) 144 000 (g) 1.17 (h) 0.777

(i) 0.0186 (j) 0.0582 (k) 0.004 88 (l) 6.47×10^{-13}

6. 4130 cm^3 7. 59.7 cm 8. 2390 g 9. 13 10. 16.6 m 11. 1.23 cm^3

12. (a) 5.56×10^{-12} (b) 4.27 (c) 0.0634 (d) 5575 (e) 0.0693
(f) 4.69 (g) 1 540 000 (h) 11 400 (i) 133 000 (j) 2 780 000

13. (a) 5.654 (b) 2.370 (c) 1.245 (d) 2.262 (e) 18.09 (f) 16 250
(g) 1332 (h) 216 000 (i) 1389 (j) 0.026 76 (k) 0.1887 (l) 0.002 375

Exercise 7-6

1. (a) 0.7202 (b) 0.5403 (c) 0.6314 (d) 0.9165 (e) 0.8525 (f) 0.9420
(g) 0.6946 (h) 0.3856 (i) 0.4393 (j) 0.5855 (k) 0.0682 (l) 0.4393
2. (a) 1 (b) 2 (c) 1 (d) 3 (e) −2 (f) −1 (g) 0 (h) 1
(i) −2 (j) −1 (k) 1 (l) 4 (m) 3 (n) 2 (o) −3 (p) 0
3. (a) 0 + .7202 (b) 1 + .4116 (c) 6 + .6284 (d) 0 + .8280
(e) 0 + .4116 (f) 0 + .8837 (g) 4 + .6551 (h) 0 + .3385
(i) 1 + .8041 (j) 3 + .7226 (k) 5 + .5145 (l) 3 + .8837
4. (a) 0 + .7936 (b) 0 + .8181 (c) 1 + .3288 (d) 2 + .6616
(e) 6 + .7876 (f) 5 + .6078 (g) 1 + .6249 (h) 1 + .3649
(i) 4 + .4807 (j) 0 + .4560 (k) 6 + .5128 (l) 1 + .5745
5. (a) 0 + .4108 (b) −1 + .6852 (c) −2 + .8792 (d) −3 + .6319
(e) 1 + .1229 (f) −3 + .6248 (g) 2 + .0738 (h) −1 + .0738
(i) 0 + .0738

Exercise 7-7

1. (a) log 25 + log 35 (b) log 75 + log 15 (c) log 35 + log 17
(d) log 25 + log 25 (e) log 15.2 + log 3.54 (f) log 3125 + log 41.27
(g) log 4.27 + log 5.83 (h) log 42.7 + log 10.2 (i) log 325 + log 1.4
(j) log 38.51 + log 21.7
2. (a) 5 + .6367 (b) 2 + .3628 (c) 3 + .1332 (d) −5 + .9483
(e) −7 + .5882 (f) 1 + .1593 (g) 7 + .6832 (h) −2 + .1836
(i) −1 + .2403
3. (a) 19.69 (b) 496.6 (c) 0.3182 (d) 169.3 (e) 52.73 (f) 7703
(g) 0.7313 (h) 751.7 (i) 649.3
4. (a) 2.035×10^8 (b) 1548 (c) 0.019 61 (d) 2000 (e) 16 840
(f) 0.001 392 (g) 16.40 (h) 2.510 (i) 153.7
5. (a) 0.6181 (b) 81.84 (c) 324.0 (d) 692.1 (e) 12.24 (f) 1.47
6. 5676 m^2 7. 0.4027 m^3 8. 5.562 g 9. 30.52 m^3 10. 1.433 g

Exercise 7-8

1. (a) log 30 − log 7 (b) log 40 − log 9 (c) log 1 − log 5 (d) log 7.5 − log 7.5
(e) log 47 − log 35 (f) log 85 − log 126 (g) log 3.6 − log 12.8
(h) log 52.7 − log 12.3 (i) log 0.5 − log 0.075 (j) log 31.26 − log 54.85
2. (a) 2 + .3216 (b) 7 + .4310 (c) 9 + .5389 (d) −1 + .4207 (e) 2 + .1261
(f) −9 + .4615 (g) −3 + .3740 (h) −3 + .8911 (i) 2 + .7495
3. (a) 13.77 (b) 17.78 (c) 1.804 (d) 1.661 (e) 130.2 (f) 1.447
(g) 2.723 (h) 0.2185 (i) 6.757
4. (a) 1.711 (b) 0.5791 (c) 0.5410 (d) 1.785 (e) 0.026 54 (f) 0.6410
(g) 0.1073 (h) 19.14 (i) 15 080
5. (a) 1.719 (b) 201.2 (c) 0.041 07 (d) 2.133 (e) 3.856 (f) 1.476
6. (a) 0.3252 (b) 5.467 (c) 0.014 65 (d) 2823
7. (a) 48.57 (b) 0.03368 (c) 0.000 281 3 (d) 0.065 24 (e) 1.354 (f) 234.8
8. 2.136 9. 0.000 042 92

Exercise 7-9

1. (a) 2 log 5.5 (b) 3 log 3.75 (c) 4 log 25.5 (d) 5 log 425 (e) $\frac{1}{2}$ log 2.353
(f) $\frac{1}{4}$ log 2.47 (g) $\frac{1}{2}$ log 375 (h) $\frac{1}{3}$ log 21.65 (i) $\frac{1}{5}$ log 3.125 (j) $\frac{1}{4}$ log 0.275
2. (a) 7 + .2375 (b) −8 + .5036 (c) 1 + .1587 (d) −1 + .1422

(e) $-1 + .3566$ (f) $-1 + .6423$ (g) $-2 + .6059$ (h) $-1 + .4709$
(i) $-1 + .8809$

3. (a) 65 540 (b) 57.19 (c) 771.7 (d) 146.4
(e) 475.2 (f) 94.76 (g) 0.019 78 (h) 0.000 000 571 9
(i) 0.002 628

4. (a) 11.18 (b) 10.1 (c) 1.597 (d) 24.20 (e) 46.26 (f) 3.149
(g) 0.6519 (h) 0.137 (i) 0.1466

5. (a) 43 800 (b) 1.318 (c) 153.1 (d) 22.99 (e) 5.070 (f) 16.04
(g) 0.000 053 79 (h) 0.4665 (i) 0.4392

6. (a) 10.75 (b) 1114 (c) 9.616 (d) 0.2779 (e) 0.2437 (f) 39 470

7. \$2653 8. 14.11 cm² 9. 0.4347 cm³ 10. 2.254 s

Review Exercise

1., 2., 3., for the student

4. (a) 1.778 (b) 5.623 (c) 3.162 (d) 2.512 (e) 141.3 (f) 5 623
(g) 138 (h) 2.138

5. (a) $10^{0.5119}$ (b) $10^{0.7218}$ (c) $10^{0.9106}$ (d) $10^{0.8657}$
(e) $10^{0.8751} \times 10^{-1}$ (f) $10^{0.3365}$ (g) $10^{0.7723}$ (h) $10^{0.6365}$

6. (a) 15.44 (b) 1 268 (c) 0.8101 (d) 13.96 (e) 0.034 75 (f) 129.3
(g) 144 700 (h) 0.005 719 (i) 1.936 (j) 7.654 (k) 0.1796 (l) 0.6799

7. (a) 14.51 (b) 6.008 (c) 0.1602 (d) 1.740

8. (a) 0.015 39 (b) 0.095 37 (c) 48.56 (d) 1.272

REVIEW AND PREVIEW TO CHAPTER 8

Exercise

1. (a) $x = 70°$, $y = 110°$ (b) $a = 60°$, $b = 70°$, $c = 110°$ (c) $a = 45°$, $b = 45°$
(d) $a = 60°$, $b = 40°$, $c = 80°$, $d = 100°$ (e) $a = 70°$, $b = c = 55°$ (f) $x = 30°$

2. (a) 6 (b) 9 (c) 12 (d) 4 (e) 26 (f) 4

3. (a) 12 (b) 42.6 (c) 54.5 (d) 18.5

4. 21.3 m

5. (a) (i) $\frac{8}{17}$ (ii) $\frac{15}{17}$ (iii) $\frac{8}{15}$ (b) (i) $\frac{3}{5}$ (ii) $\frac{4}{5}$ (iii) $\frac{3}{4}$
(c) (i) $\frac{1}{2}$ (ii) $\frac{17}{20}$ (iii) $\frac{10}{17}$ (d) (i) $\frac{y}{10}$ (ii) $\frac{x}{10}$ (iii) $\frac{y}{x}$
(e) (i) $\frac{y}{r}$ (ii) $\frac{x}{r}$ (iii) $\frac{y}{x}$ (f) (i) $\frac{b}{c}$ (ii) $\frac{a}{c}$ (iii) $\frac{b}{a}$

6. (a) 1.42 (b) 0.942 (c) 52.6 (d) 12.3 (e) 16.5 (f) 84.0

Display 8

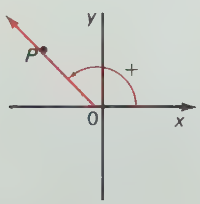
1. 16.16101523
2. 83.28742993
3. 2.152717287
4. 96.21218627
5. 16.81410039
6. 5.135701731
7. 55.35861022
8. 0.6617728623
9. 2.862859755
10. 51.44260797

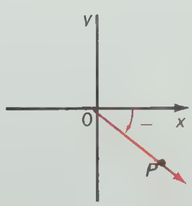
CHAPTER 8

Exercise 8-1

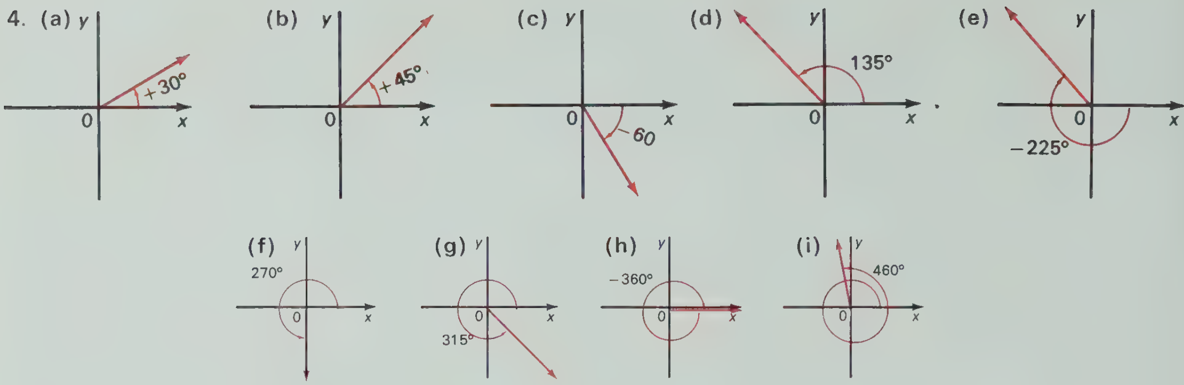
1. (b) 10 2. (a) 13 (b) 5 (c) $2\sqrt{5}$

3. (a) (b)





4.



5. (a) 120° (b) 480°, 840°, 1 200° (c) −240°, −600°, −960°

6. (a) 90° (b) 450° (c) 135° (d) −300° (e) 180° (f) −270° (g) 330° (h) −220° (i) 270° (j) −540° (k) 190° (l) 225° (m) 360° (n) 900° (o) 400° (p) 490°

Exercise 8-2

1. (i) (a) 13, (b) $\sin\theta = \frac{12}{13}$, $\cos\theta = \frac{5}{13}$, $\tan\theta = \frac{12}{5}$ (ii) (a) $\sqrt{2}$ (b) $-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1$

(iii) (a) 5, (b) $\frac{4}{5}, -\frac{3}{5}, -\frac{4}{3}$ (iv) (a) 1, (b) 1, 0, undefined

(v) (a) 2 (b) $-\frac{\sqrt{3}}{2}, \frac{1}{2}, -\sqrt{3}$ (vi) (a) 1 (b) $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -1$

2. (a) 13, $\sin\theta = -\frac{12}{13}$, $\cos\theta = \frac{5}{13}$, $\tan\theta = -\frac{12}{5}$ (b) 17, $-\frac{15}{17}, \frac{8}{17}, -\frac{15}{8}$

(c) 10, $\frac{4}{5}, -\frac{3}{5}, -\frac{4}{3}$ (d) $\sqrt{53}, -\frac{7}{\sqrt{53}}, -\frac{2}{\sqrt{53}}, \frac{7}{2}$

(e) 2, $-\frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{3}}$ (f) $3\sqrt{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1$

3. $(-4, 3), \frac{3}{5}$ 4. $(-1, -1), -\frac{1}{\sqrt{2}}$ 5. (a) 12, −12 (c) $\cos\theta = \frac{12}{13}$ or $-\frac{12}{13}$, $\tan\theta = \frac{5}{12}$ or $-\frac{5}{12}$ 6. (a) $\sin\theta = \frac{b}{\sqrt{a^2 + b^2}}$, $\cos\theta = \frac{a}{\sqrt{a^2 + b^2}}$ (b) 1 (c) $\tan\theta$

7. (a) $\sqrt{a^2 + b^2}$ (b) $\frac{a}{c}$ (c) 1 (d) equals 1 (e) $\tan\theta$ (f) equal

8.

(x, y)	$A(3, 4)$	$B(6, 8)$	$C(9, 12)$	$D(12, 16)$
r	5	10	15	20
$\sin\theta$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{4}{5}$
$\cos\theta$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$
$\tan\theta$	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$

Exercise 8-3

1. (a) (i) $r = 5$ (ii) $\csc\theta = \frac{5}{4}$, $\sec\theta = \frac{5}{3}$, $\cot\theta = \frac{3}{4}$ (b) (i) $\sqrt{2}$ (ii) $-\sqrt{2}, -\sqrt{2}, 1$
(c) (i) 5 (ii) $\frac{5}{4}, -\frac{5}{3}, -\frac{3}{4}$ (d) (i) 1 (ii) 1, undefined, 0
(e) (i) 2 (ii) $-\frac{2}{\sqrt{3}}, +2, -\frac{1}{\sqrt{3}}$ (f) (i) $\sqrt{2}$ (ii) $\sqrt{2}, -\sqrt{2}, -1$

2. (a) $r = 13$; $\csc\theta = -\frac{13}{12}$, $\sec\theta = -\frac{13}{5}$, $\cot\theta = \frac{5}{12}$ (b) 17; $\frac{17}{15}, \frac{17}{8}, \frac{8}{15}$

(c) 10; $-\frac{5}{4}, \frac{5}{3}, -\frac{3}{4}$ (d) $\sqrt{41}; -\frac{\sqrt{41}}{5}, \frac{\sqrt{41}}{4}, -\frac{4}{5}$

(e) $\sqrt{5}$; $\sqrt{5}$, $-\dfrac{\sqrt{5}}{2}$, -2 (f) $\sqrt{53}$; $\dfrac{\sqrt{53}}{7}$, $\dfrac{\sqrt{53}}{2}$, $\frac{2}{7}$

3. $(-4, 3)$, $\frac{3}{5}$ **4.** $(-1, -1)$, $-\sqrt{2}$, $-\sqrt{2}$ **5.** (a) $+5$ or -5 (c) $\sec\theta = \frac{13}{5}$ or $-\frac{13}{5}$, $\cot\theta = \frac{5}{12}$ or $-\frac{5}{12}$

6.

	r	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
(a)	13	$\frac{5}{13}$	$\frac{12}{13}$	$\frac{5}{12}$	$\frac{13}{5}$	$\frac{13}{12}$	$\frac{12}{5}$
(b)	17	$\frac{15}{17}$	$-\frac{8}{17}$	$-\frac{15}{8}$	$\frac{17}{15}$	$-\frac{17}{8}$	$-\frac{8}{15}$
(c)	17	$-\frac{8}{17}$	$-\frac{15}{17}$	$\frac{8}{15}$	$-\frac{17}{8}$	$-\frac{17}{15}$	$\frac{15}{8}$
(d)	5	$-\frac{3}{5}$	$\frac{4}{5}$	$-\frac{3}{4}$	$-\frac{5}{3}$	$\frac{5}{4}$	$-\frac{4}{3}$
(e)	10	$\frac{4}{5}$	$-\frac{3}{5}$	$-\frac{4}{3}$	$\frac{5}{4}$	$-\frac{5}{3}$	$-\frac{3}{4}$
(f)	13	$-\frac{12}{13}$	$\frac{5}{13}$	$-\frac{12}{5}$	$-\frac{13}{12}$	$\frac{13}{5}$	$-\frac{5}{12}$

7. (a) $\csc\theta = \dfrac{\sqrt{a^2 + b^2}}{b}$; $\sec\theta = \dfrac{\sqrt{a^2 + b^2}}{a}$ (b) $1 + \cot^2\theta = \csc^2 Q$ (c) $\tan\theta$

Exercise 8-4

1.

	θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
(a)	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2	$\frac{1}{\sqrt{3}}$
(b)	120°	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$\frac{2}{\sqrt{3}}$	-2	$-\frac{1}{\sqrt{3}}$
(c)	150°	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	2	$-\frac{2}{\sqrt{3}}$	$-\sqrt{3}$
(d)	210°	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\sqrt{3}$	-2	$-\frac{2}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$

2.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	$\sqrt{2}$	1
135°	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1	$\sqrt{2}$	$-\sqrt{2}$	-1
225°	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1	$-\sqrt{2}$	$-\sqrt{2}$	1
315°	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1	$-\sqrt{2}$	$\sqrt{2}$	-1

3.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
90°	1	0	und	1	und	0
180°	0	-1	0	und.	-1	und.
270°	-1	0	und	-1	und	0
360°	0	1	0	und.	1	und.

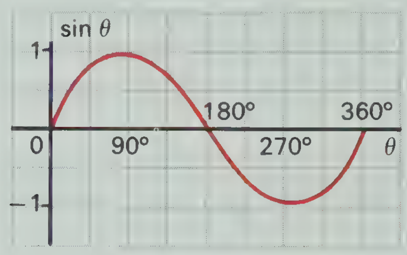
4. (a) $\frac{\sqrt{3}}{4\sqrt{2}}$ (b) $\frac{\sqrt{3}}{2\sqrt{2}}$ (c) 1 (d) $\frac{3}{16}$ 6. (a) (i) $\frac{\sqrt{3}+1}{2}$ (ii) 1 (b) no.

Exercise 8-5

1.

θ	0°	15°	30°	45°	60°	75°	- - - 360°
x	1	0.97	0.87	0.71	0.5	0.26	1
y	0	0.26	0.5	0.71	0.87	0.97	0
r	1	1	1	1	1	1	1
sin θ	0	0.26	0.5	0.71	0.87	0.97	0
cos θ	1	0.97	0.87	0.71	0.5	0.26	1
tan θ	0	0.27	0.58	1.0	1.7	3.7	0

2. (a)



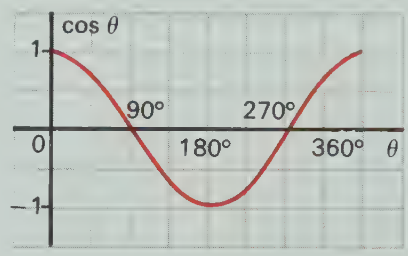
(b) 1 (c) −1

(d)

θ	0°/180°/360°	30°/150°	50°	24°/156°	210°/330°	240°/300°	215°
sin θ	0	0.5	.78	0.41	−0.5	−.87	−0.7

(e) continues, pattern repeats

3. (a)



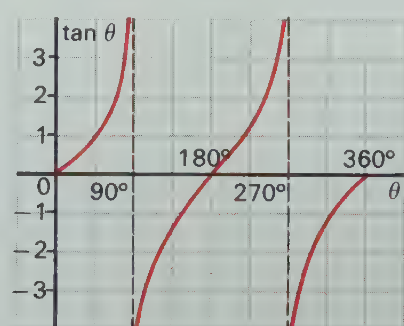
(b) 1 (c) −1

(d)

θ	90°/270°	60°/300°	50°	66°/114°	120°/240°	150°/210°	215°
cos θ	0	0.5	.64	.41	−0.5	−.87	−0.7

(e) continues, pattern repeats.

4.

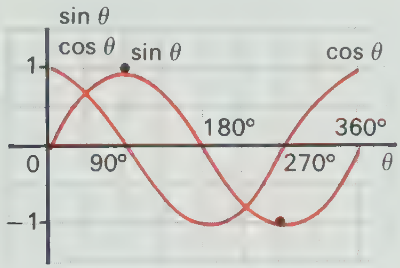


(c)

θ	0°/180°/360°	45°/225°	135°/315°	35°/215°	60°/240°	135°	330°
$\tan\theta$	0	1.0	−1.0	0.7	1.7	−1	−0.58

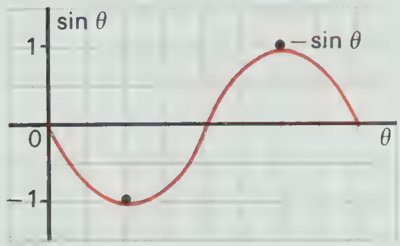
(d) continues, pattern repeats.

5. (a)

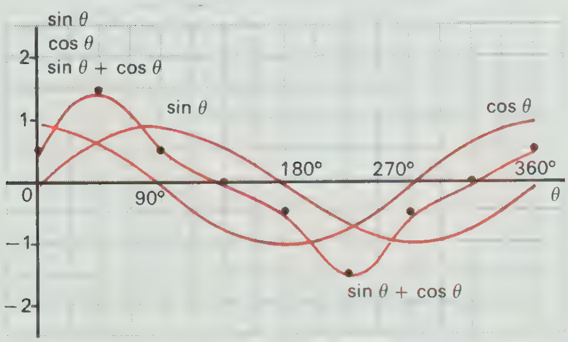


(b) 45°, 225°

6. (a)



7. (a)



(b) (i) 45° (ii) 225° (iii) 135°, 315°

Exercise 8-6

1. *Q*1 2. + 3. (a) − (b) + (c) − (d) + (e) − (f) + (g) − (h) − (i) + (j) −

Exercise 8-7

1. (a) − (b) − (c) + (d) + (e) + (f) − (g) − (h) − (i) − (j) +
2. (a) + (b) − (c) − (d) − (e) + (f) +
3. (a) 0.4067 (b) 0.3256 (c) 0.5317 (d) 0.8481 (e) 0.8481 (f) 0.5299 (g) 0.7660 (h) 0.6428 (i) 1.0000 (j) 0.6009 (k) 0.1219 (l) undefined
4. (a) 1.1792 (b) 3.8637 (c) 0.7536 (d) 1.1792 (e) 2.0627 (f) 3.2361 (g) 1.5557 (h) 1.3054 (i) undefined (j) 0.3839 (k) 3.4203 (l) 0.0000
5. (a) +0.5736 (b) −0.7431 (c) −1.4281 (d) −0.7760 (e) −0.4226 (f) 1.4281 (g) −0.5736 (h) −0.8830 (i) −0.7431 (j) −1.4281 (k) +0.7431 (l) +0.7002 (m) +0.6428 (n) −0.4663 (o) −0.9397 (p) +2.1445
6. (a) −1.1547 (b) +1.4142 (c) −2.1445 (d) −1.5557 (e) +1.0154 (f) −.4663

(g) −1.4142 (h) +1.0642 (i) +1.1547 (j) −0.4663 (k) +1.0642 (l) −0.5774

7. (a) 42°, 318° (b) 63°, 117° (c) 117°, 243° (d) 42°, 222° (e) 37°, 143°
(f) 122°, 302° (g) 166°, 194° (h) 256°, 284° (i) 78°, 258°

Exercise 8-8

1. (a) $\sin A = \frac{4}{5}$, $\cos A = \frac{3}{5}$, $\tan A = \frac{4}{3}$ (b) $\frac{8}{17}, \frac{15}{17}, \frac{8}{15}$ (c) $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1$
(d) $\frac{5}{13}, \frac{12}{13}, \frac{5}{12}$ (e) $\frac{\sqrt{3}}{2}, \frac{1}{2}, \sqrt{3}$ (f) $\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{3}}$ (g) $\frac{3}{5}, \frac{4}{5}, \frac{3}{4}$ (h) $\frac{3}{5}, \frac{4}{5}, \frac{3}{4}$

2. (a) $\csc B = \frac{5}{3}$, $\sec B = \frac{5}{4}$, $\cot B = \frac{4}{3}$ (b) $\frac{17}{15}, \frac{17}{8}, \frac{8}{15}$ (c) $\sqrt{2}, \sqrt{2}, 1$
(d) $\frac{13}{12}, \frac{13}{5}, \frac{5}{12}$ (e) $2, \frac{2}{\sqrt{3}}, \sqrt{3}$ (f) $\frac{2}{\sqrt{3}}, 2, \frac{1}{\sqrt{3}}$ (g) $\frac{5}{4}, \frac{5}{3}, \frac{3}{4}$ (h) $\frac{5}{4}, \frac{5}{3}, \frac{3}{4}$

3. (a) $\tan A = \frac{7}{8}$ (b) $\sin A = \frac{4}{\sqrt{20}}$ (c) $\tan A = 1$ (d) $\sin A = \frac{7}{10}$

4. (a) $\sec \theta = \frac{7}{4}$ (b) $\cot \theta = \frac{2}{3}$ (c) $\csc \theta = \frac{5}{4}$ (d) $\csc \theta = \sqrt{2}$

5. (a) $\sin A = \frac{4}{5}$, $\cos A = \frac{3}{5}$, $\tan A = \frac{4}{3}$, $\sin B = \frac{3}{5}$, $\cos B = \frac{4}{5}$, $\tan B = \frac{3}{4}$
$\csc A = \frac{5}{4}$, $\sec A = \frac{5}{3}$, $\cot A = \frac{3}{4}$, $\csc B = \frac{5}{3}$, $\sec B = \frac{5}{4}$, $\cot B = \frac{4}{3}$
(b) $\sin D = \frac{1}{\sqrt{17}}$, $\cos D = \frac{4}{\sqrt{17}}$, $\tan D = \frac{1}{4}$, $\sin F = \frac{4}{\sqrt{17}}$, $\cos F = \frac{1}{\sqrt{17}}$, $\tan F = \frac{4}{1}$
$\csc D = \sqrt{17}$, $\sec D = \frac{\sqrt{17}}{4}$, $\cot D = 4$, $\csc F = \frac{4}{\sqrt{17}}$, $\sec F = \sqrt{17}$, $\cot F = \frac{1}{4}$
(c) $\sin L = \frac{5}{13}$, $\cos L = \frac{12}{13}$, $\tan L = \frac{5}{12}$, $\sin K = \frac{12}{13}$, $\cos K = \frac{5}{13}$, $\tan K = \frac{12}{5}$
$\csc L = \frac{13}{5}$, $\sec L = \frac{13}{12}$, $\cot L = \frac{12}{5}$, $\csc K = \frac{13}{12}$, $\sec K = \frac{13}{5}$, $\cot K = \frac{5}{12}$
(d) $\sin P = \frac{24}{25}$, $\cos P = \frac{7}{25}$, $\tan P = \frac{24}{7}$, $\sin R = \frac{7}{25}$, $\cos R = \frac{24}{25}$, $\tan R = \frac{7}{24}$
$\csc P = \frac{25}{24}$, $\sec P = \frac{25}{7}$, $\cot P = \frac{7}{24}$, $\csc R = \frac{25}{7}$, $\sec R = \frac{25}{24}$, $\cot R = \frac{24}{7}$

6. (a) $\sin A = \frac{5}{13}$, $\cos A = \frac{12}{13}$, $\tan A = \frac{5}{12}$, $\sin C = \frac{12}{13}$, $\cos C = \frac{5}{13}$, $\tan C = \frac{12}{5}$
(b) $\csc A = \frac{13}{5}$, $\sec A = \frac{13}{12}$, $\cot A = \frac{15}{5}$, $\csc C = \frac{13}{12}$, $\sec C = \frac{13}{5}$, $\cot C = \frac{5}{12}$

7. (a) $\sin D = \frac{4}{5}$, $\cos D = \frac{3}{5}$, $\tan D = \frac{4}{3}$, $\sin F = \frac{3}{5}$, $\cos F = \frac{4}{5}$, $\tan F = \frac{3}{4}$
(b) $\csc D = \frac{5}{4}$, $\sec D = \frac{5}{3}$, $\cot D = \frac{3}{4}$, $\csc F = \frac{5}{3}$, $\sec F = \frac{5}{4}$, $\cot F = \frac{4}{3}$

8. (a) $\sin J = \frac{1}{\sqrt{2}}$, $\cos J = \frac{1}{\sqrt{2}}$, $\tan J = 1$, $\sin L = \frac{1}{\sqrt{2}}$, $\cos L = \frac{1}{\sqrt{2}}$, $\tan L = 1$
(b) $\csc J = \sqrt{2}$, $\sec J = \sqrt{2}$, $\cot J = 1$, $\csc L = \sqrt{2}$, $\sec L = \sqrt{2}$, $\cot L = 1$

9. (a) $\sin A = \frac{a}{c}$, $\cos A = \frac{b}{c}$, $\tan A = \frac{a}{b}$, $\sin B = \frac{b}{c}$, $\cos B = \frac{a}{c}$, $\tan B = \frac{b}{a}$
(b) 1 (c) $\frac{\sin A}{\cos A} = \tan A$ (d) (i) sine (ii) sine

Exercise 8-9

1. (a) 48° (b) 58° (c) 23° 2. (a) 3.35 (b) 1.65 (c) 2.24

3. (a) $a = 5$, $\angle B = 37°$, $\angle C = 53°$ (b) $a = 0.800$, $\angle A = 28°$, $\angle B = 62°$
(c) $b = 41.7$, $\angle A = 60°$, $\angle C = 30°$ (d) $\angle A = 54°$, $c = 1.38$, $b = 2.34$
(e) $\angle C = 50°$, $b = 6.04$, $c = 7.20$ (f) $\angle A = 33°$, $\angle B = 57°$, $c = 4.41$
(g) $\angle B = 19°$, $b = 0.326$, $c = 0.946$ (h) $\angle A = 24°$, $\angle C = 66°$, $b = 6.39$

Exercise 8-10

1. (a) 178 cm (b) 3.05 cm (c) 721 cm (d) 1398 cm (e) 326 cm (f) 10.9 m
(g) 7.1 m (h) 191 cm

2. (a) $\angle B = 55°$, $BC = 150$ cm, $AB = 84$ cm (b) $\angle D = 37°$, $\angle F = 53°$, $DF = 41$ cm
(c) $\angle G = 46°$, $\angle H = 44°$, JH 10 cm (d) $\angle M = 30°$, $KL = 1.0$ cm, $KM = 2.0$ cm

Exercise 8-11

1. 227 m 2. 147 m 3. 177 km 4. (a) 18 m (b) 11.5 m
5. 160 m 6. 5° 7. (a) 62.9 (b) 4 m 8. (a) 303 m (b) 195 m
9. 69.8 m 10. 2 780 m 11. 571 m 12. 6.39 km 13. 87.2 m 14. 8.6 km

15. 178 m **16.** 0.541 cm **17.** 8.55 cm **18.** 3.75 cm, 6.50 cm **19.** 21° **20.** 14°, 28°
21. (a) 6° (b) 12.56 cm **22.** 10 cm **23.** 10.4 cm, 11.6 cm **24.** 0.1499 m **25.** 64°
26. 3.21 cm **27.** 68° **28.** 3.46 cm **29.** 0.382 cm **30.** 8.274 cm

Exercise 8-12

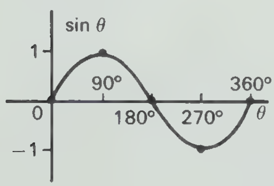
1. (a) 5.22 cm (b) 180 cm (c) 8.72 cm
2. (a) 31 cm (b) 11.9 cm (c) 9.2 cm (d) 6.07 cm (e) 7.6 cm
3. (a) $\angle C = 70°$, $\angle A = 40°$, $AB = AC = 14.6$ cm (b) $\angle A = 66°$, $b = 6.4$ cm, $c = 10$ cm
   (c) $\angle B = 21°$, $b = 4.5$ cm, $c = 12$ cm (d) $\angle A = 69°$, $b = 16.7$ cm, $a = 20.9$ cm
   (e) $\angle A = 51°$, $b = 4.56$ cm, $c = 9.82$ cm
4. 271 km, 209 km **5.** 5.43 km, 6.64 km **6.** 600 m **7** 4.24 m, 18.69 m
8. (a) 88.6 cm (b) $4.43 **9.** 10.5 km, 1.4 km **10.** 6.19 and 7.60 n.mi.

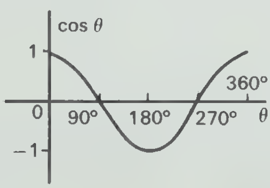
Exercise 8-13

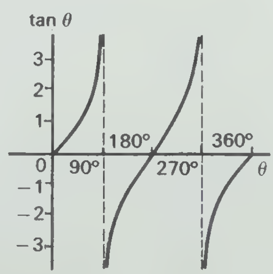
1. (a) 5.79 cm (b) 11.7 cm (c) 14.8 cm
2. (a) 3.92 cm (b) 83.52 cm (c) 83° (d) 115°

Review Exercise

1. (a) $\sin\theta = \frac{15}{17}$, $\cos\theta = \frac{8}{17}$, $\tan\theta = \frac{15}{8}$ (b) $\frac{12}{13}, -\frac{5}{13}, -\frac{12}{5}$
   (c) $-\frac{5}{\sqrt{26}}, \frac{1}{\sqrt{26}}, -5$ (d) $-\frac{\sqrt{3}}{2}, \frac{1}{2}, -\sqrt{3}$
2. (a) $\csc\theta = -\frac{13}{5}$, $\sec\theta = \frac{13}{12}$, $\cot\theta = -\frac{5}{12}$ (b) $\sqrt{2}, -\sqrt{2}, -1$
   (c) $\frac{\sqrt{53}}{7}, \frac{\sqrt{53}}{2}, \frac{2}{7}$ (d) $-2, -\frac{2}{\sqrt{3}}, \sqrt{3}$
3. $(-4, 3)$, $\frac{3}{5}$
4. (a) −0.9397 (b) −0.8660 (c) +0.8391 (d) +0.7660 (e) −0.5000
   (f) −1.7321 (g) −0.3420 (h) +0.6428 (i) −0.8391 (j) −0.7660
   (k) −0.9397 (l) −1.7321 (m) +1.1547 (n) −1.7321 (o) −2.000
   (p) +1.0353 (p) −1.0642 (r) −1.3054 (s) −0.3640 (t) −1.0000
5. (a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{2}$ (e) $\frac{1}{2}$ (f) $\frac{1}{2}$ (g) 1 (h) 1
6. (a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{2}}$ (c) 1 (d) 2 (e) 1 (f) 2
7. (a) (b) (c)







8. (a) 4.70 cm, 8.83 cm (b) 16.5 cm, 20.4 cm
   (c) 100 cm, 173 cm (d) 1.0 m, 1.0 m
9. (a) 36° (b) 52° (c) 21° (d) 30°
10. (a) $\angle B = 36°$, $c = 9.7$ m, $\angle C = 54°$ (b) $\angle A = 67°$, $\angle C = 23°$, 13 m
    (c) $\angle B = 50°$, $a = 84.6$ cm, $b = 99.6$ cm (d) $\angle C = 40°$, $AC = 21.5$ m, $BC = 28.0$ m
11. 6.34 m **12.** 11° **13.** 367 m **14.** (a) 35° (b) 4.69 m
15. (a) $\angle A = 80°$, $b = 1.58$ cm, $a = 2.72$ cm (b) $\angle B = 87°$, $b = 4.47$, $c = 2.57$ cm
    (c) $\angle A = 75°$, $b = 23.3$, $c = 32.8$ (d) $\angle A = 5°$, $a = 11.0$ cm, $a = 1.06$ cm

REVIEW AND PREVIEW TO CHAPTER 9

Exercise 1

1. 60° **2.** 30° **3.** $x = 80°$, $y = 20°$ **4.** 90° **5.** $x = 60°$, $y = 70°$ **6.** 60°
7. $x = y = 40°$ **8.** $x = 100°$, $y = 80°$ **9.** $x = 30°$, $y = 15°$

Exercise 2

1. 10 **2.** 17 **3.** 7 **4.** 24 **5.** 30 **6.** 25

Display 9

1. 9.890922255
2. 0.7060948124
3. 158.5207665
4. 0.4183372428
5. 11.55602897
6. 67.31076163
7. 39.85319868
8. 38.2746119
9. 48.81943046
10. 59.45464236

CHAPTER 9

Exercise 9-1

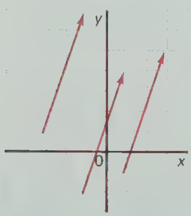
1. [7, 0], [4, 2], [−6, 5], [−4, −2] **2.** (2, 5), (2, 11), (3, 3), (−1, 3)
3. **(a)** [−3, −2] **(b)** [−1, 5] **(c)** [5, 8] **(d)** [0, −5] **(e)** [5, −2]
4. **(a)** [3, 6] **(b)** [12, 3] **(c)** [10, −3] **(d)** [3, −5] **(e)** [−3, 5]
5. **(a)** (6, 8) **(b)** (0, 0) **(c)** (4, 1) **(d)** (7, 11) **(e)** (−1, 1)
6. **(b)** and **(c)**
7. **(a)** [5, 9] **(b)** [3, 7] **(c)** [4, −1] **(d)** [8, 6]
8. **(a)** 5 **(b)** 13 **(c)** 13 **(d)** 17 **(e)** 25 **(f)** 17 **(g)** 10 **(h)** 5

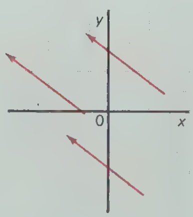
Exercise 9-2

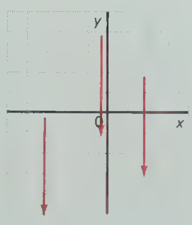
1. **(a)** [2, 4] **(b)** [−3, 5] **(c)** [−3, 0] **(d)** [0, 5] **(e)** [0, 0]
2. $\overrightarrow{AB} = [5, -2]$, $\overrightarrow{BA} = [-5, 2]$, $\overrightarrow{BC} = [-10, -8]$, $\overrightarrow{CD} = [8, 5]$, $\overrightarrow{DA} = [-3, 5]$
$\overrightarrow{AC} = [-5, -10]$, $\overrightarrow{DB} = [2, 3]$
3. **(a)** parallelogram **(b)** $\overrightarrow{PQ} = [-2, -8]$, $\overrightarrow{QR} = [11, 6]$, $\overrightarrow{SR} = [-2, -8]$, $\overrightarrow{PS} = [11, 6]$
4. **(a)** same direction **(b)** different magnitudes **(c)** increase by 75 kn
5. **(b)** [−400, 200] **(c)** 447, 27° **6.** **(b)** 13 kn **7.** [100, 173]
8. **(a)** [0, 500] **(b)** [100, 0] **(c)** [−100, 173] **9.** **(b)** 0°, 8.66

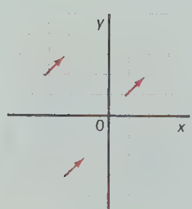
Exercise 9-3

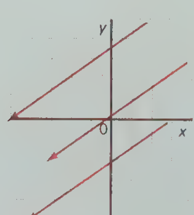
1. **(a)** (i) [4, 3] (ii) [5, 2] (iii) [6, 2] (iv) [4, 3] (v) [4, 3] **(b)** $\overrightarrow{AB}$, $\overrightarrow{GH}$, $\overrightarrow{KL}$
2. **(a)** **(b)** **(c)** **(d)** **(e)**











3. $A(-1, 4)$, $B(2, 3)$, $C(4, 6)$, $D(2, 1)$, $E(-3, -1)$, $F(4, -1)$, $G(6, 4)$ 4. $(6, 3)$
5. (a) $(-1, -6)$ (b) $(-1, -6)$ 6. (a) $(-10, -4)$ (d) no, only one.

7.

	Algebraic Vector	Magnitude	Slope
CD	$[3, -4]$	5	$-\frac{4}{3}$
EF	$[2, 4]$	$2\sqrt{5}$	2
GH	$[6, -4]$	5	$-\frac{4}{3}$
IJ	$[1, 2]$	$\sqrt{5}$	2
KL	$[1, 2]$	$\sqrt{5}$	2
MN	$[3, -4]$	5	$-\frac{4}{3}$
PQ	$[-1, -2]$	$\sqrt{5}$	2

8. (a) $a = 5$, $b = 3$ (b) $a = c$, $b = d$ (c) $[a, b] = [c, d]$

Investigation 9-4

1. (b) $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ (c) $\overrightarrow{AB} = [200, 0]$, $\overrightarrow{BC} = [300, 0]$, $\overrightarrow{AC} = [500, 0]$
2. (a) $\overrightarrow{PQ} = [0, -200]$, $\overrightarrow{QR} = [0, 300]$ (c) $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}$
3. (a) $\overrightarrow{OB}$ (b) $\vec{a} + \vec{b}$ (d) triangle 4. (a) $[4, 5]$ (b) sum (c) triangle
5. (a) $[5, 2]$ (b) $[5, 6]$ 6. (a) $[10, 2]$ (b) double (c) $3\vec{a} = \vec{b}$
7. (a) $[2, 3]$, a vector (b) $[a + c, b + d]$, a vector (c) yes, closure
8. (a) = (b) = (c) yes 9. (a) = (b) = (c) yes
10. (a) (i) $[4, 3]$ (ii) $[-7, -2]$ (iii) $[-4, 2]$ (iv) $[-3, -5]$
    (b) (i) $[a, b]$ (ii) $[a, b]$ (c) $[0, 0]$ (d) $[0, 0]$
11. (a) back to initial point (b) $[0, 0]$
12. (a) $[-2, -3]$ (b) $[-4, -11]$ (c) $[4, -7]$ (d) $[3, 5]$ (e) $[-a, -b]$
13. (c) (i) same magnitude (ii) opposite direction

Exercise 9-4

1. (a) $[7, 11]$ (b) $[4, 8]$ (c) $[1, 1]$ (d) $[5, -5]$ (e) $[-5, -7]$
   (f) $[4, 12]$ (g) $[0, 0]$ (h) $[10, -10]$ (i) $[2, 5]$ (j) $[3, 6]$
2. (a) $[-3, 7]$ (b) $[-2, 6]$ (c) $[0, 0]$ (d) $[1, 11]$ (e) $[0, 0]$ (f) $[3, 5]$
3. (a) 13 km, N23°E (b) 15 km, N (c) 16 km, N53° W
5. (a) $[5, 7]$ (b) $[9, 4]$ (c) $[4, 1]$ (d) $[1, 1]$ (e) $[-1, 0]$ (f) $[-8, -2]$
6. (a) $[1, 3]$ (b) $[-1, 1]$

Investigation 9-5

1. (b) $\vec{c} = [5, 12]$ (c) $\vec{a} = \vec{c}$ (d) 13, 26 units (e) $2\vec{a}$ (f) $\vec{b} = 2\vec{a}$
2. (a) $[9, 12]$ (b) 15, 5 units (c) 3 times
3. (a) $[-7, -9]$ (b) (i) same, (ii) opposite (c) $\vec{a} = \vec{b}$ (d) $\vec{0}$, zero vector
4. (a) $[6, 4]$, $[9, 6]$, $[-3, -2]$, $[-9, -6]$
5. (a) $[ax, bx]$ (b) $[2ax, 2bx]$ (c) $[-ax, -bx]$ (d) $[\frac{1}{2}ax, \frac{1}{2}bx]$
6. (a) magnitude increased to three times, same direction
   (b) magnitude increased to two times, opposite direction
   (c) magnitude decreased to $\frac{1}{4}$, same direction
   (d) magnitude increased to k times, same direction
   (e) magnitude increased to k times, opposite direction

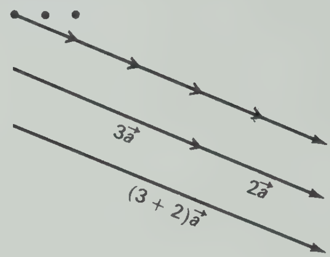
Exercise 9-5

1. (a) $[8, 14]$ (b) $[18, -3]$ (c) $[3, 6]$ (d) $[8, 20]$ (e) $[-2, 4]$
   (f) $[0, 0]$ (g) $[0, -8]$ (h) $[-1, 4]$ (i) $[21, 42]$

2. (a) [18, 11] (b) [1, 8] (c) [4, 2] (d) [−1, −1] (e) [−31, −29]
(f) [18, −12] (g) [4, 36] (h) [0, 0] (i) $[12\frac{1}{2}, 5\frac{1}{2}]$

3. (a) $[8x, 10y]$ (b) $[x, 0]$ (c) $[12x, 16y]$

4. (a)



Exercise 9-6

1. (a) [1.2] (b) [9, −9] (c) [4, 9] (d) [−3, −2] (e) [0, 0]
(f) [10, 24] (g) [5, 9] (h) $[a, 3]$ (i) $[3a, 3b]$ (j) $[x-c, y-d]$

2. (a) [8, 1] (b) [14, −20] (c) [−2, 0] (d) [−21, 5] (e) [−11, 8]
(f) [−40, 40] (g) [6, −18] (h) [6, 21]

3. (a) and (b) 4. (a) and (b) for the student (use graph paper) 5. [0, 0]

6. (a) [3, 3] (b) [−5, 19] (c) [10, −31] (d) [5, 20] (e) [5, −5]
(f) [3, −15] (g) [21, −21] (h) [8, −4] (i) [13, 36] (j) [17, 14]

Exercise 9-7

1. (a) 10.3, 61° (b) 9.22, 283° (c) 5, 128° (d) 14.1, 225°
(e) 19.7, 60° (f) 19.7, 150° (g) 15.5, 315° (h) 10, 0°

2. (a) [10.9, 5.07] (b) [−90.6, −42.2] (c) [0.985, 0.174] (d) [0, 0]
(e) [−1.00, 1.73] (f) [−17.7, −17.7] (g) [−10, 0] (h) $[m\cos\theta, m\sin\theta]$

3. (a) [5, 0] (b) [−106, 106] (c) [0, 500] (d) [7.49, 18.5]
(e) [96.1, −55.5] (f) [−3.0, −5.2]

4. (a) [100, 0], [0, 173] (b) [100, 173]

5. (a) [10, 0], [5.14, 6.13] (b) [15.14, 6.13] (c) [20, 0] (d) [4.86, −6.13]

6. (b) 44.7, 63° 7. [161, 192] 8. (a) [424, 424] (b) [467, 467] (c) [382, 382]

9. [−1.732, 3.000]

Exercise 9-8

1. 10.8, 22° 2. 325, N22°W 3. 15.8, 18° 4. 400.5, N6°E 5. 605, S83°E 6. 30.4, 10° 7. 403, N84°W 8. S56°W, 360 9. S9°E, 304 10. N85°W, 401 11. N8°W, 346 12. N81°E, 295.8 13. 130 V, 22° 14. 121 V, 23° 15. 79.1 V

Exercise 9-9

1. (a) $\vec{x}+\vec{y}$ (b) $\vec{x}-y$ (c) $2x+\vec{y}$ (d) $2\vec{x}-\vec{y}$

2. (a) $\vec{x}$ (b) $\vec{y}$ (c) $-\vec{x}$ (d) $\vec{x}+\vec{y}$ (e) $-\vec{x}$ (f) $\vec{y}-\vec{x}$
(g) $\vec{x}-\vec{y}$ (h) $\vec{x}+\vec{y}$ (i) $\frac{1}{2}(\vec{x}+\vec{y})$ (j) $\vec{x}+\vec{y}$ (k) $\vec{y}-\vec{x}$ (l) $\frac{1}{2}(\vec{y}-\vec{x})$
(m) $\frac{1}{2}(\vec{y}-\vec{x})$ (n) $\frac{1}{2}(x-\vec{y})$

3. $\overrightarrow{AD} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}$ 4. (a) $\overrightarrow{PQ} = \overrightarrow{SR}$ (b) parallelogram (c) $\overrightarrow{PS} = \overrightarrow{QR}$
(d) parallelogram

5. (a) $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{BC}$ (c) The line joining the midpoints of the sides of a triangle is parallel to and equal to one half of the third side. 6. (c) in half 7. (b) parallelogram (c) If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

Review Exercise

1. (a) (8, −3) (b) (0, 0) (c) (4, −3) (d) (8, −6) (e) (9, −2) (f) $(x+4, y-3)$

2. (a) [12, 6] (b) [8, −1] (c) [13, 23] (d) [5, −3] (e) 4.47
(f) 7.62 (g) 9.57 (h) [−6, −19]

3. (a) $\overrightarrow{CD}$ and $\overrightarrow{JK}$, $\overrightarrow{EF}$ and $\overrightarrow{NP}$ (b) $\overrightarrow{AB}$ and $\overrightarrow{DC}$, $\overrightarrow{AD}$ and $\overrightarrow{BC}$, $\overrightarrow{BE}$ and $\overrightarrow{ED}$

4. (b) $\overrightarrow{OA} = [4, 2]$, $\overrightarrow{AB} = [3, 3]$, $\overrightarrow{BC} = [-5, -5]$, $\overrightarrow{CD} = [-8, 1]$, $\overrightarrow{OD} = [-6, 1]$,
$\overrightarrow{AC} = [-2, -2]$

5. (a) [2, 10] (b) [8, 33] (c) [−6, −4] (d) [6, 1] (e) [−4, 15]
(f) [14.39] (g) [51.0] (h) <

6. (b) (i) $P(4, 2)$, $Q(8, 4)$, $R(15, 9)$, $S(17, 9)$, $T(11, 10)$
(ii) $P(2, 6)$, $Q(6, 8)$, $R(13, 13)$, $S(15, 13)$, $T(9, 14)$

7. $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{BA} = \overrightarrow{CD}$ 8. [26.9, 170] 9. 13, S67°E 10. (a) $D(6, 0)$
(b) $\overrightarrow{AB} = \overrightarrow{DC} = [2, 5]$, $\overrightarrow{BC} = \overrightarrow{AD} = [8, 2]$ (c) parallelogram 11. $x = 2$, $y = -1$

12. (a) 5, 37° (b) 5, 53° (c) 13.0, 123° (d) 8.6, 234°
(e) 17, 332° (f) $\sqrt{a^2 + b^2}$, $\tan\theta = \frac{b}{a}$

13. (a) S81°E, 253 (b) N81°E, 253 14. (b) parallelogram 15. for the student

REVIEW AND PREVIEW TO CHAPTER 10

Exercise

1. $10.00 2. 176.4 N 3. 138.8 N 4. 570 780 J 5. 58 640 6. 12.5 m
7. 32.4 8. 148 ℓ 9. $1.82 10. 591 ℓ 11. 787.5 g 12. 35 ℓ 13. 61.4 ℓ
14. 133.3 g

Display 10

1. 1977.9449
2. 26.012465
3. 1.21234
4. 94544.93
5. 1175.3451
6. 2.774313
7. 20954.23
8. 0.00362853
9. 57168773
10. 5060.7161

CHAPTER 10

Exercise 10-1

1. 375 N 2. 0.15 kg 3. 2.5 m/s² 4. 12 kg 5. 9.8N/kg

Exercise 10-2

1. (a) 6 N right (b) 4 N right (c) 7 N right (d) 4 N right (e) 28 N left
(f) 25 N right (g) 24 N right (h) 9 N left (i) 8 N left (j) 0
2. (a) 8.4 N (b) 4.7 N (c) 4.4 N (d) 4.6 N (e) 6.6 N (f) 3.8 N
4. (b) 185 N (c) 31° 5. (b) 11 600 (c) 12° 6. (b) 172 N (c) 11°

Exercise 10-3

1. (a) R = 28 N right, E = 28 N left (b) R = 8 N left, E = 8 N right
(c) R = 15 N right, E = 15 N left (d) R = 8 N right, E = 8 N left
(e) R = 4 N right, E = 4 N left
2. (a) 6.2 N (b) 2.8 N (c) 3.6 N (d) 2.9 N
3. (a) 13.2 N (b) 18.1 N (c) 33.8 N (d) 50 N (e) 22.9 N
(f) 5.10 N (g) 11.5 N (h) 19.3 N
4. 10 030 N 5. 2070 N 6. 132 N

Exercise 10-4

1. (a) 50 N, 54° (b) 25 N, 54° (c) 13 N, 67° (d) 13.5 N, 17°
(e) 15.7 N, 37° (f) 14.4 N, 57° (g) 12.5 N, 76° (h) 8.43 N, 82°
(i) 20.3 N, 47°
2. 14.1 N, 45° 3. 85 N 4. 6290 N 5. 181 N, 163° to the 90 N force
6. (a) 8920 N (b) 6° 7. (a) 14 600 N (b) 30° 8. 72.1 N 9. 44.7 N, 89.4 N

10. 372 N, 54° with respect to the 220 N force

Exercise 10-5

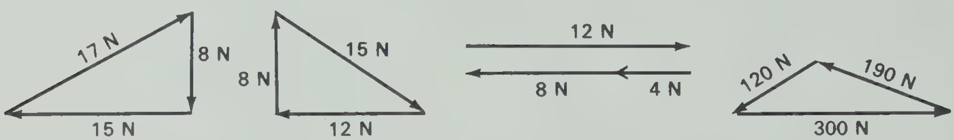
1. (a) yes (b) yes (c) yes (d) no (e) yes (f) no
2. 104°, 133°, 123 ° 3. 90°, 113°, 157° 4. 0°, 180° 5. 111°, 128°, 121°

Exercise 10-6

1. (a) 342 N, 940 N (b) 260 N, 150 N (c) 86.8 N, 492 N (d) 386 N, 460 N
   (e) 77 N, 92 N
2. (a) 164 N (b) 115 N 3. 193 N, 230 N 4. 376 N, 137 N 5. 134 N, 112 N
6. (a) 86.6 N (b) 50 N 7. (a) 239 N (b) left (c) 106 N left

Review Exercise

1. (a) 111 N (b) 49 N (c) 1250 N (d) 850 N
2. (a) 398 N (b) 47.2 N (c) 70.7 N (d) 86.6 N (e) 14.5 N (f) 25.1 N
3. 271 N 4. 70.7 N 5. 20 N left 6. for the student 7. (a) 14.4 N, 34° (b) 14 N, 30°
   (c) 72.1 N, 34° (d) 103 N, 18° (e) 103 N, 50° (f) 155 N, 42° 8. 92.2 N
9. (a) (b) (c) (d)



10. (a) 32.1 N, 38.3 N (b) 25.3 N, 54.4 N (c) 27.5 N, 47.6 N
    (d) 50 N, 86.6 N (e) 0 N, 100 N (f) 76.6 N, 64.3 N
11. 173 N, 100 N 12. 1300 N

REVIEW AND PREVIEW TO CHAPTER 11

Exercise 1

1. (a) 6 (b) 25 (c) 24 (d) 5 (e) 1 (f) 5
   (g) 5 (h) 6 (i) $\sqrt{34}$ (j) $\sqrt{15}$
2. (a) 5.7 (b) 2.9 (c) 15 (d) 7.6 (e) 9.8 (f) 25
   (g) 19 (h) 11 (i) 2.4 (j) 20

Exercise 2

1. (a) CA (b) XYZ (c) COB (d) Area QRT (e) OA
   (f) $\angle AOB$ or $\angle AOC$ (g) Area AOB (h) Area QYR (i) AEC (j) CDB
   (k) $PQRS$
2. (a) 6.0 cm (b) 13 cm (c) 48 cm (d) 3.0 cm (e) 14 cm (f) 10.6 cm
3. (a) $x = 3$ cm, $y = 3$ cm (b) $x = 9$ cm, $y = 18$ cm (c) $x = 2$ cm, $y = 3$ cm
4. (a) $x = 26$ cm, $y = 24$ cm (b) $x = 12$ cm, $y = 16$ cm (c) $x = 5$ cm, $y = 5$ cm

Exercise 3

1. (a) $x - 2y - 7 = 0$ (b) $3x + 5y - 32 = 0$ (c) $y = -6x - 4$
   (d) $y = 4x + \frac{2}{3}$ (e) $x + 2y - 9 = 0$ (f) $7x - y - 37 = 0$
   (g) $6x - 3y - 5 = 0$ (h) $y = \frac{2}{3}x + 6$ (i) $2x + y = 6$ (j) $y = 4$
2. AB, $x = 3$; BC, $y = 2$; CD, $x = 10$; AD, $y = 7$
3. QR, $x - 10y + 24 = 0$; RP, $x + y - 9 = 0$; PQ, $4x - 7y + 30 = 0$
4. $6x + y - 32 = 0$ 5. $5x - 2y - 22 = 0$ 6. $7x + 4y - 1 = 0$

Display 11

1. 29.92681162
2. 2.993937799
3. 984.5751376
4. 543.7468565
5. 3690.743049
6. 61349.02134
7. 0.4794070389
8. 3998.619129
9. 501.7751786
10. 12.186185

CHAPTER 11

Exercise 11-1

1. (a) $x^2 + y^2 = 25$ (b) $x^2 + y^2 = 81$ (c) $x^2 + y^2 = 16$ (d) $x^2 + y^2 = 144$ (e) $x^2 + y^2 = 49$ (f) $x^2 + y^2 = 64$ (g) $x^2 + y^2 = 3$ (h) $x^2 + y^2 = 2$ (i) $x^2 + y^2 = a^2$ (j) $x^2 + y^2 = r^2$
2. (a) (0, 0), 4 (b) (0, 0), 10 (c) (0, 0), 5 (d) (0, 0), $\sqrt{5}$ (e) (0, 0), 6 (f) (0, 0), 7 (g) (0, 0), $2\sqrt{3}$ (h) (0, 0), r
3. (a) 4 (b) $x^2 + y^2 = 16$
4. (a) 13 (b) $x^2 + y^2 = 169$
5. (a) $x^2 + y^2 = 25$ (b) $x^2 + y^2 = 625$ (c) $x^2 + y^2 = 10$ (d) $x^2 + y^2 = 49$
6. (a) – (ii) (b) – (v) (c) – (i) (d) – (iii) (e) – (iv)
7. (possible solutions) (a) (1, 4), (–1, 4) . . . (b) (5, 5), (–5, 5) . . . (c) (4, 2), (–4, 2) . . . (d) (9, 0), (0, –9) . . .

Exercise 11-3

1. (a) $\sqrt{10}$ (b) 5 (c) 10 (d) $\sqrt{82}$ (e) $\sqrt{(h-1)^2 + (k-2)^2}$ (f) $\sqrt{(x-h)^2 + (y-k)^2}$
2. (a) $x^2 + (y-3)^2 = 16$ (b) $(x-1)^2 + (y-2)^2 = 64$ (c) $(x-4)^2 + y^2 = 36$ (d) $(x-3)^2 + (y-6)^2 = 49$ (e) $(x-2)^2 + (y-4)^2 = 81$ (f) $(x-5)^2 + (y-2)^2 = 144$ (g) $(x-5)^2 + (y-2)^2 = 5$ (h) $(x-1)^2 + (y-5)^2 = 6$ (i) $(x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 = \frac{9}{4}$ (i) $(x-h)^2 + (y-k)^2 = r^2$
3. (a) (4, 7), 11 (b) (–5, 2), 5 (c) (2, 3), 6 (d) (2, –3), t
5. (a) $(x-4)^2 + (y-5)^2 = 9$ (b) $(x+1)^2 + (y-4)^2 = 25$ (c) $(x-3)^2 + (y+4)^2 = 49$ (d) $(x-6)^2 + y^2 = 18$
6. (a) $\sqrt{41}$ (b) $(x-5)^2 + (y+3)^2 = 41$
7. $(x-1)^2 + (y+3)^2 = 45$
8. (a) $x^2 + y^2 - 6y - 7 = 0$ (b) $x^2 + y^2 - 2x - 4y - 59 = 0$ (c) $x^2 + y^2 - 8x - 20 = 0$ (d) $x^2 + y^2 - 6x - 12y - 4 = 0$ (e) $x^2 + y^2 - 4x - 8y - 61 = 0$ (f) $x^2 + y^2 - 10x - 4y - 115 = 0$ (g) $x^2 + y^2 - 10x - 4y + 24 = 0$ (h) $x^2 + y^2 - 2x - 10y + 20 = 0$ (i) $4x^2 + 4y^2 - 4x - 4y - 7 = 0$ (j) $x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$
9. $x^2 + y^2 - 6x + 10y + 29 = 0$
10. (a) 5 units (b) $x^2 + y^2 - 6x - 4y - 12 = 0$ (c) (i) (0, 0) (ii) 5 units (iii) $x^2 + y^2 = 25$
11. (a) $x^2 + y^2 = 9$ (b) (4, –7), 3 (c) $(x-4)^2 + (y+7)^2 = 9$

Exercise 11-4

1. (a) $\frac{1}{3}$ (b) $-\frac{4}{7}$ (c) $\frac{2}{5}$ (d) undefined (e) undefined (f) $-\frac{1}{2}$ (g) –4 (h) 3 (i) undefined (j) undefined (k) 0 (l) 0 (m) $\frac{2}{3}$ (n) –1
2. (a) $y - 4 = \frac{1}{5}(x-3)$ (b) $y + 2 = 2(x-1)$ (c) $y = -\frac{2}{3}(x+4)$ (d) $y - 4 = \frac{4}{3}(x-5)$ (e) $y + 2 = 6(x-5)$ (f) $x = -3$ (g) $y = 7$ (h) $y + 1 = -4(x+2)$ (i) $x = -1$ (j) $y = 0$ (k) $y = 6$ (l) $x = 3$
3. (a) $3x + 4y - 25 = 0$ (b) $x - 3y + 10 = 0$ (c) $x = 3$ (d) $3x - 2y - 26 = 0$ (e) $y = -7$ (f) $2x + y - 5 = 0$
4. (a) $x - 2y = 0$ (b) $2x - 3y - 18 = 0$ (c) $y = \frac{2}{3}x + 2$ (d) $x = 4$ (e) $x + y - 9 = 0$ (f) $y = 5$

5. (a) ± 3 (c) $2x + 3y = 13$, $2x - 3y = 13$
6. (a) $x - 5y = 26$, $x + 5y = -26$ (b) (i) $y = 4$, $y = -4$
(ii) $x = 4$, $x = -4$
7. $3x + \sqrt{2}y = 11$ 8. $x + 2y = 0$
9. (a) -2 (b) $y = -2x$ (c) $x = \pm 2$ (d) $x = 2$, $y = -4$; $x = -2$, $x = 4$
(e) $x - 2y - 10 = 0$, $x - 2y + 10 = 0$ 10. $3x - 5y \pm 34 = 0$

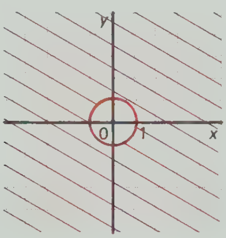
Exercise 11-5

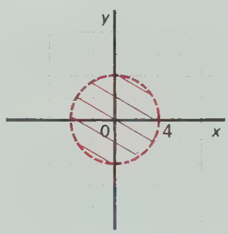
1. (a) $(0, 0)$, 8 (b) $(0, 0)$, $2\sqrt{5}$ (c) $(3, 5)$, 5 (d) $(2, -6)$, 6
(e) $(0, -2)$ $4\sqrt{2}$ (f) $(-7, 8)$, 10
2. (a) 4 (b) 11 (c) 5 (d) 13 (e) $5\sqrt{5}$ (f) 10
3. $\sqrt{10}$ 4. $2\sqrt{3}$ 5. 4 6. 12 8. 8 units 9. 3 units 10. $\sqrt{5}$, $PA:AC = 1:1$

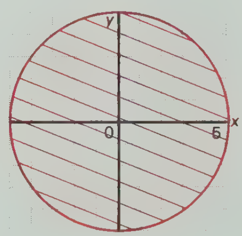
Exercise 11-6

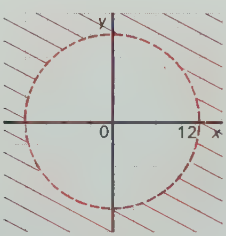
1. (b) $2x - y = 0$ 2. (a) undefined (b) 0 (c) perpendicular
3. (a) $(7, -1)$ (b) $x - y - 8 = 0$ (c) The perpendicular bisector of the chord passes through the centre.
4. $x^2 + (y - 2)^2 = 13$ 5. $(x + 2)^2 + (y - 3)^2 = 13$ 6. $x^2 + y^2 - 8x + 6y = 0$
7. (a) ± 13, (b) $(\pm 13, 0)$ (c) On the circle
(d) $-5, \frac{1}{5}$ (e) Perpendicular (f) Right angle
8. (a) Diameter (b) $\frac{1}{7}, -7$ (c) Right angle
9. (a) $(1, 2)$ (b) $(7, 5)$ (c) $\frac{1}{2}$ (d) -2 (e) Perpendicular
(f) $(3, 3)$ (g) Perpendicular bisector

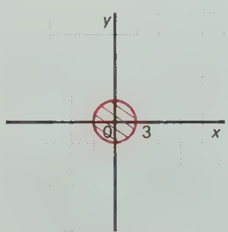
Review Exercise

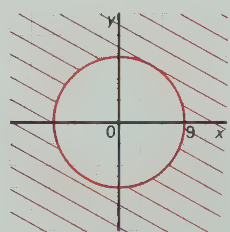
1. (a) $x^2 + y^2 = 36$ (b) $x^2 + y^2 = 81$ (c) $x^2 + y^2 = 25$
(d) $x^2 + y^2 = 25$ (e) $x^2 + y^2 = 49$ (f) $x^2 + y^2 = 2$
2. (a) $x^2 + y^2 - 2x - 4y - 4 = 0$ (b) $x^2 + y^2 - 4x - 6y - 12 = 0$
(c) $x^2 + y^2 - 8y - 21 = 0$ (d) $x^2 + y^2 - 2y - 15 = 0$
(e) $x^2 + y^2 - 14x + 4y = 0$ (f) $x^2 + y^2 - 8x - 9 = 0$
3. (a)


(b)


(c)


(d)


(e)


(f)


4. (a) $x^2 + y^2 = 5$ (b) $x + 2y - 5 = 0$
5. (a) $x^2 + y^2 = 13$ (b) $3x - 2y - 13 = 0$, $3x - 2y + 13 = 0$
6. (a) $2x - 5y + 29 = 0$ (b) $y = 6$ (c) $3x + 2y + 7 = 0$ (d) $3x - y - 11 = 0$
7. (a) $4\sqrt{15}$ (b) 7 (c) 4 (d) 5
8. (b) $C(0, 0)$, $k(4, 0)$ (c) $I(5, 0)$ (d) Slope $CI = 0$, Slope $KI = 0$ (e) collinear
9. (a) $7\sqrt{2}$ (b) $(\frac{1}{2}, -\frac{1}{2})$ (c) -1 (d) 1 10. 16 11. $x^2 + y^2 + 2x - 10y + 13 = 0$

REVIEW AND PREVIEW TO CHAPTER 12

Exercise 1

1. $0.30 2. $5.07 3. $16.02 4. $385.60 5. $2.74
6. $45.18 7. $2.15 8. $67.97 9. $67.68 10. $42.88
11. $57.06 12. $3.20 13. $103.68 14. $59.35 15. $22.89
16. $40.25 17. $247.50 18. $49.38

Exercise 2

1. 0.0904 2. 0.1726 3. 0.2548 4. 0.4110 5. 0.1644
6. 0.5753 7. 0.2301 8. 0.4986 9. 0.1644 10. 0.2466
11. 0.3288 12. 0.0630 13. 0.1014 14. 0.1534 15. 0.8219
16. 2.4932 17. 0.0822 18. 0.1233 19. 0.0055 20. 0.0082

Exercise 3

1. 69 d, 0.1890 a 2. 105 d, 0.2877 a 3. 141 d, 0.3863 a
4. 198 d, 0.5425 a 5. 193 d, 0.5288 a 6. 119 d, 0.3260 a
7. 158 d, 0.4329 a 8. 45 d, 0.1233 a 9. 121 d, 0.3315 a
10. 135 d, 0.3699 a 11. 137 d, 0.3753 a 12. 68 d, 0.1863 a
13. 03-08 14. 05-06 15. 05-23 16. 08-18
17. 10-06 18. 10-07 19. 01-21 20. 05-14

Exercise 4

2. $35.70 4. $0.44 6. $0.90 8. $0.38 10. $1.00 12. $1.57 14. $8.60

Display 12

1. 6754.979953
2. 19.213128
3. 67149.78368
4. 1.5988805
5. 0.0180504604
6. 22490.19717
7. 2879028.675
8. 67404202.42
9. 13959.84163
10. 0.7277489643

CHAPTER 12

Exercise 12-1

1. (b) $I = Prt$, $A = I + P$, $A = P(1 + rt)$ 2. $r = \frac{I}{Pt}$, $t = \frac{I}{Pr}$
3. (a) $140, $4140 (b) $10.26, $530.26 (c) $6.31, $646.31
(d) $44.38, $684.38 (e) $3.16, $323.16 (f) $143.83, $2643.83
4. (a) $6 (b) $66 (c) $5200 (d) $750 (e) 8% (f) 12%
(g) 152 d (h) 1 a
5. (a) $377.47 (b)$546.09 (c) $5613.29 (d) $127.32 (e) $431.92
(f) $7597.13
6. $10000 7. $4857.27 8. 5.5% 9. $1687.50 10. $557.32 11. $512.49
12. $1536.31, 08-25 13. (a) $5.42 (b) 8% 14. 07-04

Exercise 12-2

1. (a) 10% (b) 10% (c) 12% (d) $10\frac{1}{2}$% (e) $3\frac{1}{2}$% (f) 10%
2. 14.7% both 3. 10% 4. (a) $69.67 (b) 10.8% 5. 10.6%

Exercise 12-3

1. (a) $6.50, 20% (b) $87.05, 24.2% (c) $37.50, 5.3%
(d) $365.50, 12.4% (e) $16.80, 30.8% (f) $78.10, 9.8%
2. (a) $60.50 (b) 15.4% 3. 12.4% 4. $7.00, 9.2% 5. $35 6. $8.77 7. $298.95
8. (a) $512 (b) $22 9. 534.75 10. $1650, $150, 9.7% 11. $43 400, $39 223
12. $339.06, $94.64

Exercise 12-4

1. \$36 2. \$27 3. \$20.25, 18% 5. \$35.91, 12% 6. \$54, 18% 7. (a) \$33.75, (b) \$39.38 (c) \$45.00 8. (a) \$39.84, $12\frac{1}{4}$% (b) \$39, 12%

Exercise 12-5

1. (a) 04-14 (b) \$1200 (c) that it would not be paid
2. Mr. Wilson \$7500
   Mr. Marshall $8\frac{1}{2}$%
   03-23 \$57.63
   04-25 \$7557.64
3. (a) 02-13, \$276.86 (b) 05-14, \$542.48 (c) 06-29, \$2797.66
   (d) 05-24, \$733.68 (e) 07-08, \$5814.46 (f) 11-20, \$7662.43
   (g) 10-07, \$181.06 (h) 10-30, \$301.61 (i) 11-30, \$5262.95
   (j) 01-02, \$1813.02

Exercise 12-6

1. (a) 06-27, \$646.72 (b) 07-11, \$1480.58 (c) 06-16, \$832.67
   (d) 11-29, \$263.32 (e) 12-28, \$1245.68 (f) 07-07, \$2485.31
2. (a) \$350.43 (b) \$603.18 (c) \$1501.28 (d) \$200.33 (e) \$923.68 (f) \$150.86
3. \$5759.73 4. 8.7% 5. 50 d, \$565.82 6. \$2561.06

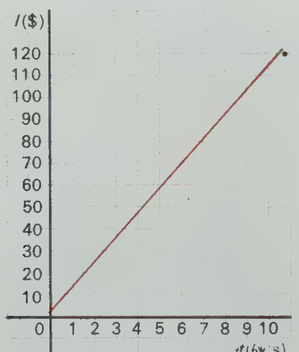
Exercise 12-7

1. $P = \frac{I}{rt}$, $r = \frac{I}{Pt}$, $t = \frac{I}{Pr}$
2. (a) \$40 (b) \$2.50 (c) \$200 (d) \$500 (e) 10% (f) 12%
   (g) 2 a (h) $\frac{1}{2}$ a
3. (a) Straight line (b) 80 (c) 0 (d) \$80, \$160, \$240, \$400
4. (a) Curve asymptotic to the positive axes (b) P approaches zero
   (c) P gets very large (d) \$2500, \$1000, \$500

Review Exercise

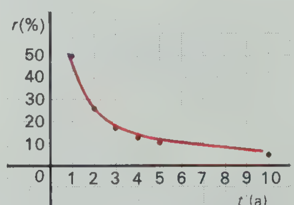
1. (a) \$267.09 (b) \$1604.48 (c) \$772.19 2. 1014% 3. \$8.57 4. (a) \$57.41 (b) 10.15% 5. (a) 10.4% versus 10.5% (b) \$324.24 versus \$486.24 6. 7.7% 7. \$21.00, 9% 8. \$72.25 9. \$273.08 10. \$1185.85 11. (a) \$507.64 (b) \$501.19 12. \$769.18, \$761.12

13.



$I = Prt$
$I = 12t \quad P = \$100,\ r = 0.12$

14.



REVIEW AND PREVIEW TO CHAPTER 13

Exercise

1. (a) \$5.00 (b) \$9.00 (c) \$12.00 (d) \$8.00 (e) \$10.50 (f) \$7.25
   (g) \$16.75 (h) \$12.50 (i) \$4.00 (j) \$2.50 (k) \$9.00 (l) \$27.00

2. (a) \$106.00 (b) \$108.00 (c) \$109.00 (d) \$107.00 (e) \$112.00
(f) \$103.00 (g) \$102.00 (h) \$102.33 (i) \$103.50 (j) \$101.00
3. (a) 7 (b) 42 (c) 10 (d) 23 (e) 51 (f) 100
4. (a) (i) 1.0, 1.1, 1.2, 1.3 (ii) 1.0, 1.1, 1.21, 1.33
(iii) 3.0, 3.3, 3.6, 3.9 (iv) 3.0, 3.3, 3.63, 3.99
(b) (i) 1.0, 1.3, 1.6, 1.9 (ii) 1.0, 1.34, 1.79, 2.40
(iii) 10, 13, 16, 19 (iv) 10, 13.4, 17.9, 24.0

Display 13

1. 4.2798
2. 158.2666665
3. 109.3125
4. 80.4375
5. 209.63324
6. 1068.7905
7. 573.9115
8. 7428.75
9. 230.092825
10. 206.24875

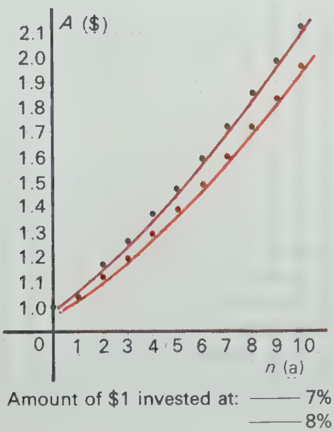
CHAPTER 13

Exercise 13-1

1. (a) 5% (b) 12% (c) 4% (d) 3.5% (e) 3% (f) 4.5%
(g) 2% (h) 3.75% (i) 1% (j) 1.5%
2. (a) \$200 (b) \$240 (c) \$375 (d) \$5 (e) \$360 (f) \$202.50
3. \$131.24, \$631.24 4. \$115.93 5. (a) \$6700.50 (b) \$6500 6. \$3555.25
7. \$337.46

Exercise 13-2

1. (a) $i = 0.06$, $n = 12$ (b) $i = 0.045$, $n = 32$ (c) $i = 0.08$, $n = 10$
(d) $i = 0.035$, $n = 10$ (e) $i = 0.01$, $n = 36$ (f) $i = 0.055$, $n = 18$
(g) $i = 0.025$, $n = 18$ (h) $i = 0.015$, $n = 33$
2. (a) 1.061 36 (b) 1.601 03 (c) 4.660 96 (d) 1.857 49 (e) 5.516 02
(f) 4.048 93 (g) 1.384 23 (h) 5.852 36 (i) 2.952 16 (j) 2.432 54
(k) 1.132 80 (l) 1.851 94 (m) 3.869 68 (n) 1.384 23 (o) 2.158 93
3. (a) \$136.05 (b) \$1772.20 (c) \$180.61 (d) \$17 908.50
(e) \$131.21 (f) \$851.33 (g) \$114 117 (h) \$1 030 000
4. (a) \$141.30 (b) \$34.21 (c) \$1126.10 (d) \$837.68 (e) \$217.25
(f) \$1429.50 (g) \$1013.49 (h) \$10 272.72 (i) \$3488.77 (j) \$1011.19
5. (a) \$816.48 (b) \$143.11 (c) \$855.46 (d) \$11 260.95
(e) \$11 416.65 (f) \$4453.53 (g) \$59 807.40 (h) \$145 409.00
6. (a) 9 a (b) 9 a (c) 8.75 a 7. 7%
8. (a) \$638.55 (b) \$704.00 (c) \$65.45
9. (a) (i) \$1402.55 (ii) \$1967.15 (b) (i) \$1469.33 (ii) \$2158.93
(c)



Exercise 13-3

1. (a) \$587.73 (b) \$631.14 (c) \$89.71 (d) \$1377.02 (e) \$112.68
2. \$2458.52 3. \$43 292.00 4. \$77 648.50, \$65 113.00 5. (a) \$5310.60 (b) \$5791.80 6. \$3794.32 7. \$6242.33 8. \$1 403.83 9. \$898.19 10. \$15
11. \$2011.81

Exercise 13-4

1. (a) \$4354.92 (b) \$134.53 (c) \$3226.57 (d) \$134.46
2. (a) \$1242.60 (b) \$1548.74 (c) \$134.42 (d) \$270.11
3. \$1188.57 4. \$517.73 5. \$403.02

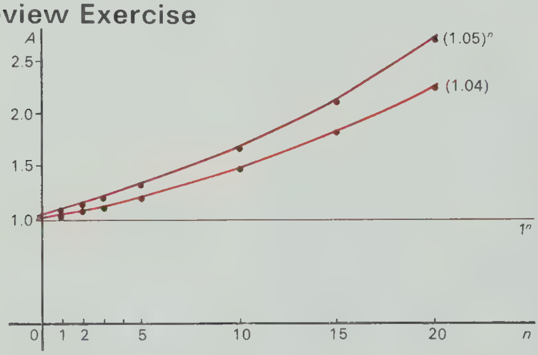
Exercise 13-5

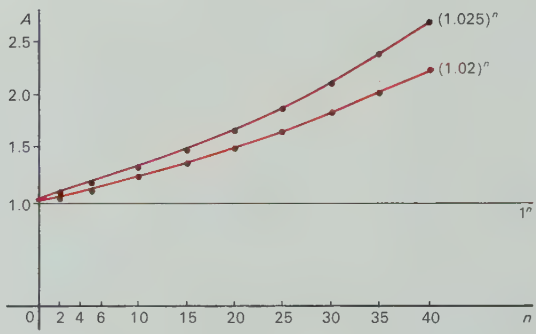
1. (a) 12 (b) 20 (c) 23 (d) 21 (e) 31 (f) 37
2. (a) 1.6% (b) 3.5% (c) 6.0% (d) 5.1% (e) 3.4% (f) 7.6%
3. 14 4. 12 5. 8 a 10 mos. 6. 4% 7. 7.5% 8. 9.7% 9. \$330.62 10. 4.7 a
11. 5.3%

Exercise 13-6

1. (a) \$2498.40 (b) \$142.09 (c) \$141.76 (d) \$2264.45
2. (a) \$205.03 (b) \$330.13 (c) 6219.60 (d) \$3387.29
3. \$1804.95 4. The first by \$24.29 5. \$1017.97 6. (a) \$7842.59 (b) (i) \$9532.75 (ii) \$12 774.72 7. \$3863.58 8. (a) The second (b) By \$322.73 9. \$3695.18
10. \$930.86 11. (a) \$767.90 (b) \$2208.48 12. \$4781.57

Review Exercise





1. (a)

	n	0	1	2	3	5	10	15	20
A	$(1.04)^n$	1	1.04	1.0816	1.1249	1.2167	1.4802	1.8009	2.1911
	$(1.05)^n$	1	1.05	1.1025	1.1576	1.2763	1.6289	2.0789	2.6533

(b)

	n	0	2	5	10	15	20	25	30	35	40
A	$(1.02)^n$	1	1.0404	1.1041	1.2190	1.3459	1.4860	1.6406	1.8114	1.9999	2.2084
	$(1.025)^n$	1	1.0506	1.1314	1.2801	1.4483	1.6386	1.8539	2.0976	2.3732	2.6851

(c) \$56 approx. \$181 approx. (d) \$457 approx.

2. (a) \$4302.41 (b) \$110.46 (c) \$1389.16 (d) \$2950.22 (e) \$12 942.05 (f) \$1406.77
3. (a) \$813.50 (b) \$2717.85 (c) \$1025.76 (d) \$1024.19 (e) \$393.79 (f) \$270.32

4. (a) $593.29 (b) $114.24 (c) $2074.73 (d) $511.40
(e) $640.39 (f) $846.29
5. 10 a 6. 6.2% 7. $426.47 8. 8 a, 9 mo. 9. The first by $3.98
10. $4889.80 11. $1987.96 12. $451.97 13. (a) $307.60 (b) $18.92

REVIEW AND PREVIEW TO CHAPTER 14

1. (a) $603.01 (b) $1448.63 (c) $2184.50 (d) $4873.51 (e) $1930.86
(f) $1387.90 (g) $2014.96 (h) $1227.41 (i) $10 546.28 (j) $2273.79
2. (a) $503.21 (b) $222.32 (c) $221.21 (d) $862.91 (e) $180.84
(f) $61.69 (g) $388.12 (h) $207.73 (i) $186.29 (j) $728.08
3. (a) $39.85 (b) $3.73 (c) $27.10 (d) $12.61 (e) $90.19
(f) $80.63 (g) $216.61 (h) $514.62 (i) $576.15 (j) $220.53
4. (a) $61.04 (b) $32.30 (c) $181.33 (d) $628.67 (e) $61.37
(f) $8.6% (g) 13.1% (h) 8.6% (i) 1.3% (j) 3.8%
5. (a) $2.90 (b) $3.25 (c) $8.17 (d) $10.02 (e) $85.64 (f) $74.16
(g) $64.44 (h) $63.64 (i) $623.06 (j) $442.80 (k) $189.48 (l) $476.27

Display 14

1. 22489.28774
2. 6753.339599
3. 94699.23672
4. 119077.6799
5. 109477.8026
6. 4557.354062
7. 1064.322531
8. 3754.47119
9. 3966.964422
10. 2418.284192

CHAPTER 14

Exercise 14-1

1. (a) $62.50, $82.50, $20.00 (b) $5.50, $8.75, $48.75 (c) $0.12, $12.96, $4.32
(d) $295.99, $795.90, $887.97 (e) $28.00, $648.00, −$88.00
2. (a) $2062.80 (b) $171.90 3. $7750.00 4. $5888 5. $310.54
6. (a) $2266.51 (b) 91% (c) (i) 91% (ii) 182% (iii) 364%

Exercise 14-2

1. (a) $31 015 (b) $132 812 (c) $11 025 (d) $9056
2. (a) $1064 (b) $1393 (c) $15 250 (d) $3220
3. $151 4. $265 200 5. (a) $144 565 (b) $108 435 (c) $51 235
6. (a) $19 120 (b) 21% 7. (a) $12 520, $3660 (b) 18.5%

Exercise 14-3

1. Gross profit is given first:
(a) $1908.48 (32%), $834.96 (14%) (b) $19 756.80 (28%), $12 700.80 (18%)
(c) $388 340.00 (42%), $129 452.00 (14%) (d) $3054.72 (37%), $1238.40 (15%)
(e) $5154.60 (33%), $2343.00 (15%)

Exercise 14-4

1. (a) $23.58 (b) $32.70 (c) $2.20 (d) $41.00 (e) $142.00 (f) $21.20
2. (a) $36.25 (b) $50.10 (c) $31.10 (d) $52.05 (e) $27.00 (f) $55.70
3. (a) $4.40 (b) $30.00 (c) $19.50 (d) $175.00 (e) $100.00 (f) $168.00
4. (a) $5.60 (b) $9.00 (c) $30.00 (d) $70.00 (e) $42.00 (f) $81.00
5. (a) 45%, $52.78 (b) $3.68, $12.88 (c) $3.50, 71% (d) $753.06, 42%
6. (a) $18.72, 32% (b) $3.99, $1.71 (c) $18.85, $32.50 (d) $1601.10, 27%
7. $3.10, 50% 8. (a) $171 400 (b) 63.4% (c) 38.8% 9. $2.00 10. $675.00

Exercise 14-5

2. (a) 32.9% (b) 47% (c) 32.4% (d) 39.4%
3. $96.90 4. $1.61, $5.11 5. $11.00, 36.4% 6. (a) 37% (b) $105.22

(c) $40 500.00 7. (a) $72 000 (b) $400 000 8. (a) $19 950 (b) $33 600

Exercise 14-6

1. (a) $69.37 (b) $176.90 (c) $471.01 (d) $1629.52 (e) $4475.29
2. $334.65 3. $3998.40, $3723.30 4. (a) $4475.29, 9.37, $9.40 (b) $292.60, $296.61
5. (a) $23.29 (b) $22.43 6. (a) $11.80 (b) $10.50 7. $58,45, no

Exercise 14-7

1. (a) 32% (b) 37% (c) 24% (d) 43% (e) 51% (f) 35.5%
2. (a) $430.26 (b) $48.94 (c) $10.17 (d) $647.19 (e) $2.92 (f) $16.16
3. 36.6%, 36.8% 4. (a) 22.3%, 23.3% (b) 38.3%, 36.5%
5. Company A items 3, 4, 6, 7, 9, 10 Total $917.41
   Company B items 1, 2, 5, 8, Total $179.81

Exercise 14-8

1. (a) 22% (b) 15% (c) 18% (d) 16% (e) $4\frac{1}{2}$%
2. 4.9% 3. 5% 4. 19.6% 5. (a) $42.72 (b) 10.3% 6. 6.9%
7. (a) 33%, 6% (b) (i) 25%, 12% (ii) 30%, 6% (iii) 21%, 8% (iv) 28%, 11%

Exercise 14-9

1. (a) $436.16 (b) $700.18 (c) $4826.34 (d) $19.29 (e) $107.23
2. $2579.36 3. $4139.52 4. (a) $508.60 (b) $518.98 (c) $527.94
5. (a) $4375.00 (b) $822.00 6. $751.78

Exercise 14-10

1. (a) $535.96 (b) $8257.20 (c) $5617.92 (d) $1002.40 (e) $8391.04
2. (a) $86.88 (b) $582.42 (c) $219.31 (d) $2.88 (e) $19.77
3. (a) $5.17
4. (a) $68.54 (b) $126.14 (c) $3231.37 (d) $23.17 (e) $311.44
   (f) $91.28 (g) $221.91 (h) $22.36

Review Exercise

1. (a) $2550.00 (b) $7800 2. (a) $9100 3. $57.30
4. $5.11 5. $6.63 6. (a) 31.6% (b) 29.6%
7. (a) 25%, 20%, 10% (b) 25%, 20%, 10% (c) $66\frac{2}{3}$%, 40%, 8% (d) 75%, 43%, 15%
   (e) 50%, $33\frac{1}{3}$%, 15%
8. $51.01 9. 38.8% 10. 18% 11. $26.21 12. $23.79 13. $214.96
14. $1.48

REVIEW AND PREVIEW TO CHAPTER 15

Exercise 1

1. (a) 50.761 (b) 753.217 (c) 569.574 (d) 969.135
2. (a) 146.52 (b) 1.563 (c) 25.78 (d) 0.024 93
3. (a) 268.65 (b) 0.682 95 (c) 154.3875 (d) 0.023 125
4. (a) 130 (b) 0.0596 (c) 136 (d) 0.0393

Exercise 2

1. (a) 9, 11, 15, 17, 20, 21, 22, 28, 46, 75, 85
   (b) 9, 24, 24, 29, 31, 33, 33, 47, 56, 83, 85, 85 (c) $\frac{5}{32}, \frac{3}{16}, \frac{1}{4}, \frac{7}{16}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}$

Exercise 3

1. (a) 62.75 (b) 62.25 (c) 55.7 (d) 506.5

Display 15

1. 19.63670033
2. 2.293933052
3. 2.216803454
4. 0.9509008173
5. 1.2711639
6. 739.2840292
7. 44.31079518
8. 94.32
9. 0.4432007477
10. 20.60255186

CHAPTER 15

Exercise 15-1

	Mean	Median	Range	Mode
1.	16.4	16	12	
2.	5.9	6	6	4
3.	7.9	8	5	8
4.	58.25	61	29	
5.	200.5	197.5	121	
6.	58.0	58.0	0.7	
7.	0.0685	0.070	0.055	0.070
8.	5.57	5.8	4.6	5.9
9.	50.6	41.1	85.6	
10.	0.62	0.705	1.00	

11. (a) $M = 18$ $\bar{x} = 18$ $R = 15$ (b) $M = 17$ $\bar{x} = 17.5$ $R = 11$
12. (a) 1.77 m (b) 0.26 m
13. (a) 85.94 kg (b) 16 kg
14. (a) 237 km/h, 207 km/h (b) 16.3 s (c) 221 km/h

Exercise 15-2

1. 20.7 cm 2. 16.8 3. $\bar{x} = 5.16$ a 5. $810 6. 59 800

Exercise 15-3

1. 21 cm 2. 16.8 pushups 3. 4.9 a
4. $700 (a grouping of 0–$300, $301–$600 . . . gives a median of $692.80)
5. Mean $0.69, Median $0.59, the $2.39 item has increased the mean more than the median since it is much greater than all other items.
6. $54 900

Exercise 15-4

1. She received 74% of the possible marks and did as well or better than 80% of the students.
2. 70%, 60th percentile.
3. (a) 60%, 45th percentile (b) 81%, 77th percentile (c) 25%, 7.5th percentile (d) 59%, 51st percentile (e) 87%, 28 percentile (f) 87%, 87th percentile
4. (a) 9th (b) 63rd (c) 112th (d) 7th (e) 9th (f) 1553rd (g) 30th (h) 49th
5. $Q_1 = 9$, $M = 11$, $Q_3 = 13$
6. (b) (i) 13th percentile (ii) 52nd percentile (iii) 94th percentile
   (c) (i) 31 (ii) 65 (iii) 9
   (d) Scores less than 52 (e) Scores greater than 88

Exercise 15-5

1. (a) 100 (b) 1.15 mm (c) 50 (d) 0.14 mm (e) 0.07 mm (f) 2.01 mm
2. (a) 8 cm (b) 1.1 cm (c) 20.7 cm 3. 18.1
4. (a) 7.2 cars (b) 3.0 cars (c) 7.5 cars
   the difference shows the distribution is not symmetric.
5. $213 (using a class interval of $300)
6. (a) $95 100 (b) $12 100 (c) The interquartile range tells us half the houses are within $12 100 of the median price. The range is unrepresentative because of the one high price of $125 000.

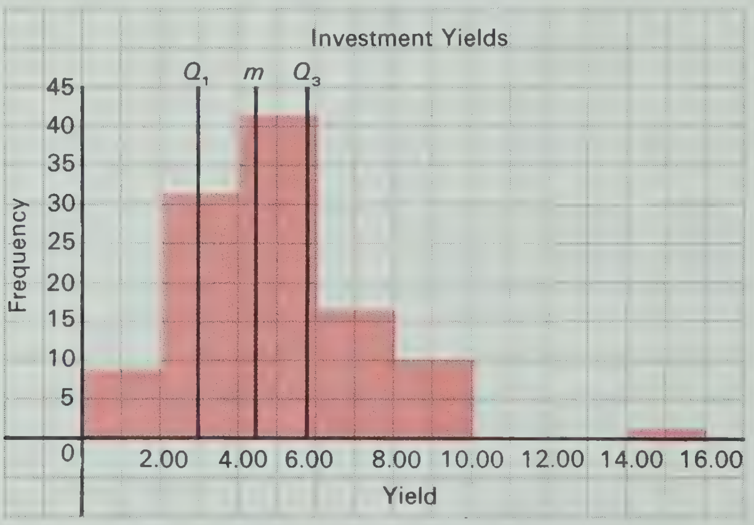
Exercise 15-6

1. (a) 666 (b) 24.94 mm–25.06 mm 2. $A - 1$, $B - 7$, $C - 17$, $D - 17$, $E - 7$, $F - 1$
3. The top values are open interval, no maximum value. 4. 3.5 5. 1.66 cm
6. (a) 20 mm (b) 0.2 mm 7. (a) 211 h (b) A–Poor, B–Worst, C–Best, D–Poor
8. $19 100
9. (a) 193.2 g (b) 206.8 g (c) 190 g (d) 210 g (e) 170 g (f) 230 g

Review Exercise

1. Mean—the "average" annual rain fall.
   Median—the middle mark of a class.
   Mode—the most frequently purchased shoe size.
2. Interquartile range—half the house prices fall within the interquartile range.
   Standard deviation—the mean strength of the fish line is 10 kg with a standard deviation of 0.2 kg.
   Range —the range of sizes of bolts to be required to assemble the machine.
3. Percentage mark gives the marks gained out of a possible 100.
   Percentile mark ranks the person with others in their group.
   The percentage mark will indicate how well the task was accomplished, the percentile mark will indicate how the person did relative to others in the group.
4. (a) $\bar{x} = 6.60$ S.D. $= 2.90$ (b) $M = 4.60$ Interquartile Range $= 1.34$

   (c)



5. (a) 51 (b) 26th percentile
6. (a) 160.3 cm (b) 169.7 cm (c) 158.5 cm (d) 171.5 cm (e) 145.5 cm
   (f) 184.5 cm
7. (b) 1.7 km (c) 1.6 km

Days Expressed in Decimal Equivalents of a Year – 365 Day Basis

For Figuring Interest, Cancellation of Insurance Premiums, Etc.

	January		February		March		April		May		June	
Day of Month	Day of Year	Decimal Equiva-lent	Day of Year	Decimal Equiva-lent	Day of Year	Decimal Equiva-lent	Day of Year	Decimal Equiva-lent	Day of Year	Decimal Equiva-lent	Day of Year	Decimal Equiva-lent
1	1	0.0027	32	0.0877	60	0.1644	91	0.2493	121	0.3315	152	0.4164
2	2	0.0055	33	0.0904	61	0.1671	92	0.2521	122	0.3342	153	0.4192
3	3	0.0082	34	0.0932	62	0.1699	93	0.2548	123	0.3370	154	0.4219
4	4	0.0110	35	0.0959	63	0.1726	94	0.2575	124	0.3397	155	0.4247
5	5	0.0137	36	0.0986	64	0.1753	95	0.2603	125	0.3425	156	0.4274
6	6	0.0164	37	0.1014	65	0.1781	96	0.2630	126	0.3452	157	0.4301
7	7	0.0192	38	0.1041	66	0.1808	97	0.2658	127	0.3479	158	0.4329
8	8	0.0219	39	0.1068	67	0.1836	98	0.2685	128	0.3507	159	0.4356
9	9	0.0247	40	0.1096	68	0.1863	99	0.2712	129	0.3534	160	0.4384
10	10	0.0274	41	0.1123	69	0.1890	100	0.2740	130	0.3562	161	0.4411
11	11	0.0301	42	0.1151	70	0.1918	101	0.2767	131	0.3589	162	0.4438
12	12	0.0329	43	0.1178	71	0.1945	102	0.2795	132	0.3616	163	0.4466
13	13	0.0356	44	0.1205	72	0.1973	103	0.2822	133	0.3644	164	0.4493
14	14	0.0384	45	0.1233	73	0.2000	104	0.2849	134	0.3671	165	0.4521
15	15	0.0411	46	0.1260	74	0.2027	105	0.2877	135	0.3699	166	0.4548
16	16	0.0438	47	0.1288	75	0.2055	106	0.2904	136	0.3726	167	0.4575
17	17	0.0466	48	0.1315	76	0.2082	107	0.2932	137	0.3753	168	0.4603
18	18	0.0493	49	0.1342	77	0.2110	108	0.2959	138	0.3781	169	0.4630
19	19	0.0521	50	0.1370	78	0.2137	109	0.2986	139	0.3808	170	0.4658
20	20	0.0548	51	0.1397	79	0.2164	110	0.3014	140	0.3836	171	0.4685
21	21	0.0575	52	0.1425	80	0.2192	111	0.3041	141	0.3863	172	0.4712
22	22	0.0603	53	0.1452	81	0.2219	112	0.3068	142	0.3890	173	0.4740
23	23	0.0630	54	0.1479	82	0.2247	113	0.3096	143	0.3918	174	0.4767
24	24	0.0658	55	0.1507	83	0.2274	114	0.3123	144	0.3945	175	0.4795
25	25	0.0685	56	0.1534	84	0.2301	115	0.3151	145	0.3973	176	0.4822
26	26	0.0712	57	0.1562	85	0.2329	116	0.3178	146	0.4000	177	0.4849
27	27	0.0740	58	0.1589	86	0.2356	117	0.3205	147	0.4027	178	0.4877
28	28	0.0767	59	0.1616	87	0.2384	118	0.3233	148	0.4055	179	0.4904
29	29	0.0795			88	0.2411	119	0.3260	149	0.4082	180	0.4932
30	30	0.0822			89	0.2438	120	0.3288	150	0.4110	181	0.4959
31	31	0.0849			90	0.2466			151	0.4137		

	July		August		September		October		November		December		
Day of Month	Day of Year	Decimal Equiva-lent	Day of Year	Decimal Equiva-lent	Day of Year	Decimal Equiva-lent	Day of Year	Decimal Equiva-lent	Day of Year	Decimal Equiva-lent	Day of Year	Decimal Equiva-lent	Day of Month
1	182	0.4986	213	0.5836	244	0.6685	274	0.7507	305	0.8356	335	0.9178	1
2	183	0.5014	214	0.5863	245	0.6712	275	0.7534	306	0.8384	336	0.9205	2
3	184	0.5041	215	0.5890	246	0.6740	276	0.7562	307	0.8411	337	0.9233	3
4	185	0.5068	216	0.5918	247	0.6767	277	0.7589	308	0.8438	338	0.9260	4
5	186	0.5096	217	0.5945	248	0.6795	278	0.7616	309	0.8466	339	0.9288	5
6	187	0.5123	218	0.5973	249	0.6822	279	0.7644	310	0.8493	340	0.9315	6
7	188	0.5151	219	0.6000	250	0.6849	280	0.7671	311	0.8521	341	0.9342	7
8	189	0.5178	220	0.6027	251	0.6877	281	0.7699	312	0.8548	342	0.9370	8
9	190	0.5205	221	0.6055	252	0.6904	282	0.7726	313	0.8575	343	0.9397	9
10	191	0.5233	222	0.6082	253	0.6932	283	0.7753	314	0.8603	344	0.9425	10
11	192	0.5260	223	0.6110	254	0.6959	284	0.7781	315	0.8630	345	0.9452	11
12	193	0.5288	224	0.6137	255	0.6986	285	0.7808	316	0.8658	346	0.9479	12
13	194	0.5315	225	0.6164	256	0.7014	286	0.7836	317	0.8685	347	0.9507	13
14	195	0.5342	226	0.6192	257	0.7041	287	0.7863	318	0.8712	348	0.9534	14
15	196	0.5370	277	0.6219	258	0.7068	288	0.7890	319	0.8740	349	0.9562	15
16	197	0.5397	228	0.6247	259	0.7096	289	0.7918	320	0.8767	350	0.9589	16
17	198	0.5425	229	0.6274	260	0.7123	290	0.7945	321	0.8795	351	0.9616	17
18	199	0.5452	230	0.6301	261	0.7151	291	0.7973	322	0.8822	352	0.9644	18
19	200	0.5479	231	0.6329	262	0.7178	292	0.8000	323	0.8849	353	0.9671	19
20	201	0.5507	232	0.6356	263	0.7205	293	0.8027	324	0.8877	354	0.9699	20
21	202	0.5534	233	0.6384	264	0.7233	294	0.8055	325	0.8904	355	0.9726	21
22	203	0.5562	234	0.6411	265	0.7260	295	0.8082	326	0.8932	356	0.9753	22
23	204	0.5589	235	0.6438	266	0.7288	296	0.8110	327	0.8959	357	0.9781	23
24	205	0.5616	236	0.6466	267	0.7315	297	0.8137	328	0.8986	358	0.9808	24
25	206	0.5644	237	0.6493	268	0.7342	298	0.8164	329	0.9014	359	0.9836	25
26	207	0.5671	238	0.6521	269	0.7370	299	0.8192	330	0.9041	360	0.9863	26
27	208	0.5699	239	0.6548	270	0.7397	300	0.8219	331	0.9068	361	0.9890	27
28	209	0.5726	240	0.6575	271	0.7425	301	0.8247	332	0.9096	362	0.9918	28
29	210	0.5753	241	0.6603	272	0.7452	302	0.8274	333	0.9123	363	0.9945	29
30	211	0.5781	242	0.6630	273	0.7479	303	0.8301	334	0.9151	364	0.9973	30
31	212	0.5808	243	0.6658			304	0.8329			365	1.0000	31

Figure 12-1

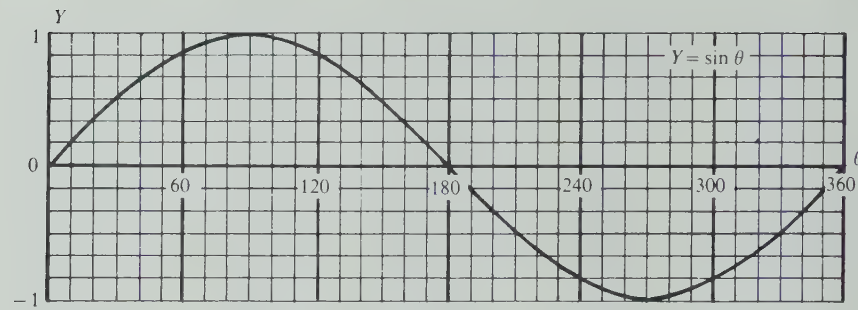
TRIGONOMETRIC RATIOS

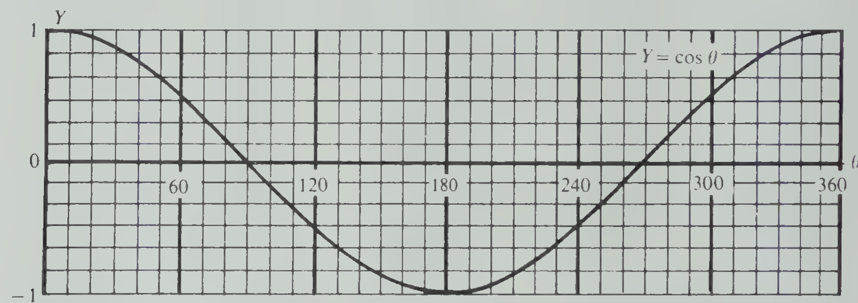
0°	sin θ	cos θ	tan θ	cot θ	sec θ	cosec θ
0	0.0000	1.0000	0.0000	—	1.0000	—
1	0.0175	0.9999	0.0175	57.290	1.0001	57.299
2	0.0349	0.9994	0.0349	28.636	1.0006	28.654
3	0.0523	0.9986	0.0524	19.081	1.0014	19.107
4	0.0698	0.9976	0.0699	14.301	1.0024	14.335
5	0.0872	0.9962	0.0875	11.430	1.0038	11.474
6	0.1045	0.9945	0.1051	9.5144	1.0055	9.5668
7	0.1219	0.9926	0.1228	8.1443	1.0075	8.2055
8	0.1392	0.9903	0.1405	7.1154	1.0098	7.1853
9	0.1564	0.9877	0.1584	6.3137	1.0125	6.3924
10	0.1737	0.9848	0.1763	5.6713	1.0154	5.7588
11	0.1908	0.9816	0.1944	5.1445	1.0187	5.2408
12	0.2079	0.9782	0.2126	4.7046	1.0223	4.8097
13	0.2250	0.9744	0.2309	4.3315	1.0263	4.4454
14	0.2419	0.9703	0.2493	4.0108	1.0306	4.1336
15	0.2588	0.9659	0.2680	3.7320	1.0353	3.8637
16	0.2756	0.9613	0.2867	3.4874	1.0403	3.6279
17	0.2924	0.9563	0.3057	3.2708	1.0457	3.4203
18	0.3090	0.9511	0.3249	3.0777	1.0515	3.2361
19	0.3256	0.9455	0.3443	2.9042	1.0576	3.0715
20	0.3420	0.9397	0.3640	2.7475	1.0642	2.9238
21	0.3584	0.9336	0.3839	2.6051	1.0711	2.7904
22	0.3746	0.9272	0.4040	2.4751	1.0785	2.6695
23	0.3907	0.9025	0.4245	2.3558	1.0864	2.5593
24	0.4067	0.9136	0.4452	2.2460	1.0946	2.4586
25	0.4226	0.9063	0.4663	2.1445	1.1034	2.3662
26	0.4384	0.8988	0.4877	2.0503	1.1126	2.2812
27	0.4540	0.8910	0.5095	1.9626	1.1223	2.2027
28	0.4695	0.8830	0.5317	1.8807	1.1326	2.1300
29	0.4848	0.8746	0.5543	1.8040	1.1433	2.0627
30	0.5000	0.8660	0.5774	1.7320	1.1547	2.0000
31	0.5150	0.8572	0.6009	1.6643	1.1666	1.9416
32	0.5299	0.8481	0.6249	1.6003	1.1792	1.8871
33	0.5446	0.8387	0.6494	1.5399	1.1924	1.8361
34	0.5592	0.8290	0.6745	1.4826	1.2062	1.7883
35	0.5736	0.8192	0.7002	1.4281	1.2208	1.7434
36	0.5878	0.8090	0.7265	1.3764	1.2361	1.7013
37	0.6018	0.7986	0.7536	1.3270	1.2521	1.6616
38	0.6157	0.7880	0.7813	1.2799	1.2690	1.6243
39	0.6293	0.7772	0.8098	1.2349	1.2867	1.5890
40	0.6428	0.7660	0.8391	1.1917	1.3054	1.5557
41	0.6561	0.7547	0.8693	1.1504	1.3250	1.5242
42	0.6691	0.7431	0.9004	1.1106	1.3456	1.4945
43	0.6820	0.7314	0.9325	1.0724	1.3673	1.4663
44	0.6947	0.7193	0.9657	1.0355	1.3902	1.4395
45	0.7071	0.7071	1.0000	1.0000	1.4142	1.4142

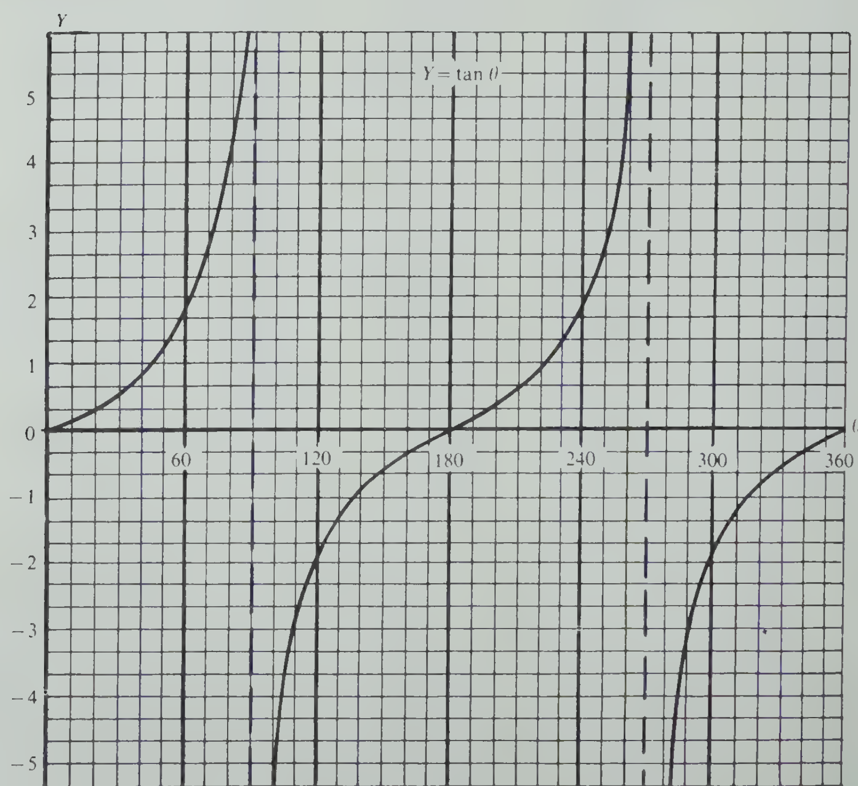
TRIGONOMETRIC RATIOS

θ°	sin θ	cos θ	tan θ	cot θ	sec θ	cosec θ
46	0.7193	0.6947	1.0355	0.9657	1.4395	1.3902
47	0.7314	0.6820	1.0724	0.9325	1.4663	1.3673
48	0.7431	0.6691	1.1106	0.9004	1.4945	1.3456
49	0.7547	0.6561	1.1504	0.8693	1.5242	1.3250
50	0.7660	0.6428	1.1917	0.8391	1.5557	1.3054
51	0.7772	0.6293	1.2349	0.8098	1.5890	1.2867
52	0.7880	0.6157	1.2799	0.7813	1.6243	1.2690
53	0.7986	0.6018	1.3270	0.7536	1.6616	1.2521
54	0.8090	0.5878	1.3764	0.7265	1.7013	1.2361
55	0.8192	0.5736	1.4281	0.7002	1.7434	1.2208
56	0.8290	0.5592	1.4826	0.6745	1.7883	1.2062
57	0.8387	0.5446	1.5399	0.6494	1.8361	1.1924
58	0.8481	0.5299	1.6003	0.6249	1.8871	1.1792
59	0.8572	0.5150	1.6643	0.6009	1.9416	1.1666
60	0.8660	0.5000	1.7320	0.5774	2.0000	1.1547
61	0.8746	0.4848	1.8040	0.5543	2.0627	1.1433
62	0.8830	0.4695	1.8807	0.5317	2.1300	1.1326
63	0.8910	0.4540	1.9626	0.5095	2.2027	1.1223
64	0.8988	0.4384	2.0503	0.4877	2.2812	1.1126
65	0.9063	0.4226	2.1445	0.4663	2.3662	1.1034
66	0.9136	0.4067	2.2460	0.4452	2.4586	1.0946
67	0.9205	0.3907	2.3558	0.4245	2.5593	1.0864
68	0.9272	0.3746	2.4751	0.4040	2.6695	1.0785
69	0.9336	0.3584	2.6051	0.3839	2.7904	1.0711
70	0.9397	0.3420	2.7475	0.3640	2.9238	1.0642
71	0.9455	0.3256	2.9042	0.3443	3.0715	1.0576
72	0.9511	0.3090	3.0777	0.3249	3.2361	1.0515
73	0.9563	0.2924	3.2708	0.3057	3.4203	1.0457
74	0.9613	0.2756	3.4874	0.2867	3.6279	1.0403
75	0.9659	0.2588	3.7320	0.2680	3.8637	1.0353
76	0.9703	0.2419	4.0108	0.2493	4.1336	1.0306
77	0.9744	0.2250	4.3315	0.2309	4.4454	1.0263
78	0.9782	0.2079	4.7046	0.2126	4.8097	1.0223
79	0.9816	0.1908	5.1445	0.1944	5.2408	1.0187
80	0.9848	0.1737	5.6713	0.1763	5.7588	1.0154
81	0.9877	0.1564	6.3137	0.1584	6.3924	1.0125
82	0.9903	0.1392	7.1154	0.1405	7.1853	1.0098
83	0.9926	0.1219	8.1443	0.1228	8.2005	1.0075
84	0.9945	0.1045	9.5144	0.1051	9.5668	1.0055
85	0.9962	0.0872	11.430	0.0875	11.474	1.0038
86	0.9976	0.0698	14.301	0.0699	14.335	1.0024
87	0.9986	9.0523	19.081	0.0524	19.107	1.0014
88	0.9994	0.0349	28.636	0.0349	28.654	1.0006
89	0.9999	0.0175	57.290	0.0175	57.299	1.0001
90	1.0000	0.0000	—	0.0000	—	1.0000

Trigonometric Graphs







The Number of Each Day of the Year

DAY OF MONTH	JAN	FEB	MAR	APR	MAY	JUN	JUL	AUG	SEP	OCT	NOV	DEC	DAY OF MONTH
1	1	32	60	91	121	152	182	213	244	274	305	335	1
2	2	33	61	92	122	153	183	214	245	275	306	336	2
3	3	34	62	93	123	154	184	215	246	276	307	337	3
4	4	35	63	94	124	155	185	216	247	277	308	338	4
5	5	36	64	95	125	156	186	217	248	278	309	339	5
6	6	37	65	96	126	157	187	218	249	279	310	340	6
7	7	38	66	97	127	158	188	219	250	280	311	341	7
8	8	39	67	98	128	159	189	220	251	281	312	342	8
9	9	40	68	99	129	160	190	221	252	282	313	343	9
10	10	41	69	100	130	161	191	222	253	283	314	344	10
11	11	42	70	101	131	162	192	223	254	284	315	345	11
12	12	43	71	102	132	163	193	224	255	285	316	346	12
13	13	44	72	103	133	164	194	225	256	286	317	347	13
14	14	45	73	104	134	165	195	226	257	287	318	348	14
15	15	46	74	105	135	166	196	227	258	288	319	349	15
16	16	47	75	106	136	167	197	228	259	289	320	350	16
17	17	48	76	107	137	168	198	229	260	290	321	351	17
18	18	49	77	108	138	169	199	230	261	291	322	352	18
19	19	50	78	109	139	170	200	231	262	292	323	353	19
20	20	51	79	110	140	171	201	232	263	293	324	354	20
21	21	52	80	111	141	172	202	233	264	294	325	355	21
22	22	53	81	112	142	173	203	234	265	295	326	356	22
23	23	54	82	113	143	174	204	235	266	296	327	357	23
24	24	55	83	114	144	175	205	236	267	297	328	358	24
25	25	56	84	115	145	176	206	237	268	298	329	359	25
26	26	57	85	116	146	177	207	238	269	299	330	360	26
27	27	58	86	117	147	178	208	239	270	300	331	361	27
28	28	59	87	118	148	179	209	240	271	301	332	362	28
29	29		88	119	149	180	210	241	272	302	333	363	29
30	30		89	120	150	181	211	242	273	303	334	364	30
31	31		90		151		212	243		304		365	31

Square Roots

n	$\sqrt{n}$	n	$\sqrt{n}$	n	$\sqrt{n}$	n	$\sqrt{n}$	n	$\sqrt{n}$
1	1.0000	41	6.4031	81	9.0000	121	11.0000	161	12.6886
2	1.4142	42	6.4807	82	9.0554	122	11.0454	162	12.7279
3	1.7321	43	6.5574	83	9.1104	123	11.0905	163	12.7671
4	2.0000	44	6.6333	84	9.1652	124	11.1355	164	12.8062
5	2.2361	45	6.7082	85	9.2195	125	11.1803	165	12.8452
6	2.4495	46	6.7823	86	9.2736	126	11.2250	166	12.8841
7	2.6458	47	6.8557	87	9.3274	127	11.2694	167	12.9228
8	2.8284	48	6.9282	88	9.3808	128	11.3137	168	12.9615
9	3.0000	49	7.0000	89	9.4340	129	11.3578	169	13.0000
10	3.1623	50	7.0711	90	9.4868	130	11.4018	170	13.0384
11	3.3166	51	7.1414	91	9.5394	131	11.4455	171	13.0767
12	3.4641	52	7.2111	92	9.5917	132	11.4891	172	13.1149
13	3.6056	53	7.2801	93	9.6437	133	11.5326	173	13.1529
14	3.7417	54	7.3485	94	9.6954	134	11.5758	174	13.1909
15	3.8730	55	7.4162	95	9.7468	135	11.6190	175	13.2288
16	4.0000	56	7.4833	96	9.7980	136	11.6619	176	13.2665
17	4.1231	57	7.5498	97	9.8489	137	11.7047	177	13.3041
18	4.2426	58	7.6158	98	9.8995	138	11.7473	178	13.3417
19	4.3589	59	7.6812	99	9.9499	139	11.7898	179	13.3791
20	4.4721	60	7.7460	100	10.0000	140	11.8322	180	13.4164
21	4.5826	61	7.8103	101	10.0499	141	11.8743	181	13.4536
22	4.6904	62	7.8740	102	10.0995	142	11.9164	182	13.4907
23	4.7958	63	7.9373	103	10.1489	143	11.9583	183	13.5277
24	4.8990	64	8.0000	104	10.1980	144	12.0000	184	13.5647
25	5.0000	65	8.0623	105	10.2470	145	12.0416	185	13.6015
26	5.0990	66	8.1240	106	10.2956	146	12.0830	186	13.6382
27	5.1962	67	8.1854	107	10.3441	147	12.1244	187	13.6748
28	5.2915	68	8.2462	108	10.3923	148	12.1655	188	13.7113
29	5.3852	69	8.3066	109	10.4403	149	12.2066	189	13.7477
30	5.4772	70	8.3666	110	10.4881	150	12.2474	190	13.7840
31	5.5678	71	8.4262	111	10.5357	151	12.2882	191	13.8203
32	5.6569	72	8.4853	112	10.5830	152	12.3288	192	13.8564
33	5.7446	73	8.5440	113	10.6301	153	12.3693	193	13.8924
34	5.8310	74	8.6023	114	10.6771	154	12.4097	194	13.9284
35	5.9161	75	8.6603	115	10.7238	155	12.4499	195	13.9642
36	6.0000	76	8.7178	116	10.7703	156	12.4900	196	14.0000
37	6.0828	77	8.7750	117	10.8167	157	12.5300	197	14.0357
38	6.1644	78	8.8318	118	10.8628	158	12.5698	198	14.0712
39	6.2450	79	8.8882	119	10.9087	159	12.6095	199	14.1067
40	6.3246	80	8.9443	120	10.9545	160	12.6491	200	14.1421

AMOUNT OF 1 $(1 + i)^n$

i n	½%	1%	1½%	2%	2½%	3%	3½%	i n
1	1.005 00	1.010 00	1.015 00	1.020 00	1.025 00	1.030 00	1.035 00	1
2	1.010 03	1.020 10	1.030 23	1.040 40	1.050 63	1.060 90	1.071 23	2
3	1.015 08	1.030 30	1.045 68	1.061 21	1.076 89	1.092 73	1.108 72	3
4	1.020 15	1.040 60	1.061 36	1.082 43	1.103 81	1.125 51	1.147 52	4
5	1.025 25	1.051 01	1.077 28	1.104 08	1.131 41	1.159 27	1.187 69	5
6	1.030 38	1.061 52	1.093 44	1.126 16	1.159 69	1.194 05	1.229 26	6
7	1.035 53	1.072 14	1.109 84	1.148 69	1.188 69	1.229 87	1.272 28	7
8	1.040 71	1.082 86	1.126 49	1.171 66	1.218 40	1.266 77	1.316 81	8
9	1.045 91	1.093 69	1.143 39	1.195 09	1.248 86	1.304 77	1.362 90	9
10	1.051 14	1.104 62	1.160 54	1.218 99	1.280 08	1.343 92	1.410 60	10
11	1.056 40	1.115 67	1.179 95	1.243 37	1.312 09	1.384 23	1.459 97	11
12	1.061 68	1.126 83	1.195 62	1.268 24	1.344 89	1.425 76	1.511 07	12
13	1.066 99	1.138 09	1.213 55	1.293 61	1.378 51	1.468 53	1.563 96	13
14	1.072 32	1.149 47	1.231 76	1.319 48	1.412 97	1.512 59	1.618 69	14
15	1.077 68	1.160 97	1.250 23	1.345 87	1.448 30	1.557 97	1.675 35	15
16	1.083 07	1.172 58	1.268 99	1.372 79	1.484 51	1.604 71	1.733 99	16
17	1.088 49	1.184 30	1.288 02	1.400 24	1.521 62	1.652 85	1.794 68	17
18	1.093 93	1.196 15	1.307 34	1.428 25	1.559 66	1.702 43	1.857 49	18
19	1.099 40	1.208 11	1.326 95	1.456 81	1.598 65	1.753 51	1.922 50	19
20	1.104 90	1.220 19	1.346 86	1.485 95	1.638 62	1.806 11	1.989 79	20
21	1.110 42	1.232 39	1.367 06	1.515 67	1.679 58	1.860 29	2.059 43	21
22	1.115 97	1.244 72	1.387 56	1.545 98	1.721 57	1.916 10	2.131 51	22
23	1.121 55	1.257 16	1.408 38	1.576 90	1.764 61	1.973 59	2.206 11	23
24	1.127 16	1.269 73	1.429 50	1.608 44	1.808 73	2.032 79	2.283 33	24
25	1.132 80	1.282 43	1.450 95	1.640 61	1.853 94	2.093 78	2.363 24	25
26	1.138 46	1.295 26	1.472 71	1.673 42	1.900 29	2.156 59	2.445 96	26
27	1.144 15	1.308 21	1.494 80	1.706 89	1.947 80	2.221 29	2.531 57	27
28	1.149 87	1.321 29	1.517 22	1.741 02	1.997 50	2.287 93	2.620 17	28
29	1.155 62	1.334 50	1.539 98	1.775 84	2.046 41	2.356 57	2.711 88	29
30	1.161 40	1.347 85	1.563 08	1.811 36	2.097 57	2.427 26	2.806 79	30
31	1.167 21	1.361 33	1.586 53	1.847 59	2.150 01	2.500 08	2.905 03	31
32	1.173 04	1.374 94	1.610 32	1.884 54	2.203 76	2.575 08	3.006 71	32
33	1.178 91	1.388 69	1.634 48	1.922 23	2.258 85	2.652 34	3.111 94	33
34	1.184 80	1.402 58	1.659 00	1.906 68	2.315 32	2.731 91	3.220 86	34
35	1.190 73	1.416 60	1.683 88	1.999 89	2.373 21	2.813 86	3.333 59	35
36	1.196 68	1.430 77	1.709 14	2.039 89	2.432 54	2.898 28	3.450 27	36
37	1.202 66	1.445 08	1.734 78	2.080 69	2.493 35	2.985 23	3.571 03	37
38	1.208 68	1.459 53	1.760 80	2.122 30	2.555 68	3.074 78	3.696 01	38
39	1.214 72	1.474 12	1.787 21	2.164 74	2.619 57	3.167 03	3.825 37	39
40	1.220 79	1.488 86	1.814 02	2.208 04	2.685 06	3.264 04	3.959 26	40

i n	4%	4½%	5%	5½%	6%	7%	8%	i n
1	1.040 00	1.045 00	1.050 00	1.055 00	1.060 00	1.070 00	1.080 00	1
2	1.081 60	1.092 03	1.102 50	1.113 03	1.123 60	1.144 90	1.166 40	2
3	1.124 86	1.141 17	1.157 63	1.174 24	1.191 02	1.225 04	1.259 71	3
4	1.169 86	1.192 52	1.215 51	1.238 82	1.262 48	1.310 80	1.360 49	4
5	1.216 65	1.246 18	1.276 28	1.306 96	1.338 23	1.402 55	1.469 33	5
6	1.265 32	1.302 26	1.340 10	1.378 84	1.418 52	1.500 73	1.586 87	6
7	1.315 93	1.360 86	1.407 10	1.454 68	1.503 63	1.605 78	1.713 82	7
8	1.368 57	1.422 10	1.477 46	1.534 69	1.593 85	1.718 19	1.850 93	8
9	1.423 31	1.486 10	1.551 33	1.619 09	1.689 48	1.838 46	1.999 00	9
10	1.480 24	1.552 97	1.628 89	1.708 14	1.790 85	1.967 15	2.158 93	10
11	1.539 45	1.622 85	1.710 34	1.802 09	1.898 30	2.104 85	2.331 64	11
12	1.601 03	1.695 88	1.795 86	1.901 21	2.012 20	2.252 19	2.518 17	12
13	1.665 07	1.772 20	1.885 65	2.005 77	2.132 93	2.409 85	2.719 62	13
14	1.731 68	1.851 94	1.979 93	2.116 09	2.260 90	2.578 53	2.937 19	14
15	1.800 94	1.935 28	2.078 93	2.232 48	2.396 56	2.759 03	3.172 17	15
16	1.872 98	2.022 37	2.182 87	2.355 26	2.540 35	2.952 16	3.425 94	16
17	1.947 90	2.113 38	2.292 02	2.484 80	2.692 77	3.158 81	3.700 02	17
18	2.025 82	2.208 48	2.406 62	2.621 47	2.854 34	3.379 93	3.996 02	18
19	2.106 85	2.307 86	2.526 95	2.765 65	3.025 60	3.616 53	4.315 70	19
20	2.191 12	2.411 71	2.653 30	2.917 76	3.207 14	3.869 68	4.660 96	20
21	2.278 77	2.520 24	2.785 96	3.078 23	3.399 56	4.140 56	5.033 83	21
22	2.369 92	2.633 65	2.925 26	3.247 54	3.603 54	4.430 40	5.436 54	22
23	2.464 72	2.752 17	3.071 52	3.426 15	3.819 75	4.740 53	5.871 46	23
24	2.563 30	2.876 01	3.225 10	3.614 59	4.048 93	5.072 37	6.341 18	24
25	2.665 84	3.005 43	3.386 35	3.813 39	4.291 87	5.427 43	6.848 48	25
26	2.772 47	3.140 68	3.555 67	4.023 13	4.549 38	5.807 35	7.396 35	26
27	2.883 37	3.282 01	3.733 46	4.244 40	4.822 35	6.213 87	7.988 06	27
28	2.998 70	3.429 70	3.920 13	4.477 84	5.111 69	6.648 84	8.627 11	28
29	3.118 65	3.584 04	4.116 14	4.724 12	5.418 39	7.114 26	9.317 27	29
30	3.243 40	3.745 32	4.321 94	4.983 95	5.743 49	7.612 26	10.062 66	30
31	3.373 13	3.913 86	4.538 04	5.258 07	6.088 10	8.145 11	10.867 67	31
32	3.508 06	4.089 98	4.764 94	5.547 26	6.453 39	8.715 27	11.737 08	32
33	3.648 38	4.274 03	5.003 19	5.852 36	6.840 59	9.325 34	12.676 05	33
34	3.794 32	4.446 36	5.253 35	6.174 24	7.251 03	9.978 11	13.690 13	34
35	3.946 09	4.667 35	5.516 02	6.513 83	7.686 09	10.676 58	14.785 34	35
36	4.130 93	4.877 38	5.791 82	6.872 09	8.147 25	11.423 94	15.968 17	36
37	4.268 09	5.096 86	6.081 41	7.250 05	8.636 09	12.223 62	17.245 63	37
38	4.438 81	5.326 22	6.385 48	7.648 80	9.154 25	13.079 27	18.625 28	38
39	4.616 37	5.565 90	6.704 75	8.069 49	9.703 51	13.994 82	20.115 30	39
40	4.801 02	5.816 36	7.039 99	8.513 31	10.285 72	14.974 46	21.724 52	40

PRESENT VALUE OF 1, $\frac{1}{(1+i)^n}$

i n	½%	1%	1½%	2%	2½%	3%	3½%	i n
1	0.995 02	0.990 10	0.985 22	0.980 39	0.975 61	0.970 87	0.966 18	1
2	0.990 07	0.980 30	0.970 66	0.961 17	0.951 81	0.942 60	0.933 51	2
3	0.985 15	0.970 59	0.956 32	0.942 32	0.928 60	0.915 14	0.901 94	3
4	0.980 25	0.960 98	0.942 18	0.923 85	0.905 95	0.888 49	0.871 44	4
5	0.975 37	0.951 47	0.928 26	0.905 73	0.883 85	0.862 61	0.841 97	5
6	0.970 52	0.942 05	0.914 54	0.887 97	0.862 30	0.837 48	0.813 50	6
7	0.965 69	0.932 72	0.901 03	0.870 56	0.841 27	0.813 09	0.785 99	7
8	0.960 89	0.923 48	0.887 71	0.853 49	0.820 75	0.789 41	0.759 41	8
9	0.956 10	0.914 34	0.874 59	0.836 76	0.800 73	0.766 42	0.733 73	9
10	0.951 35	0.905 29	0.861 67	0.820 35	0.781 20	0.744 09	0.708 92	10
11	0.946 61	0.896 32	0.848 93	0.804 26	0.762 14	0.722 42	0.684 95	11
12	0.941 91	0.887 45	0.836 39	0.788 49	0.743 56	0.701 38	0.661 78	12
13	0.937 22	0.878 66	0.824 03	0.773 03	0.725 42	0.680 95	0.639 40	13
14	0.932 56	0.869 96	0.811 85	0.757 88	0.707 73	0.661 12	0.617 78	14
15	0.927 92	0.861 35	0.799 85	0.743 01	0.690 47	0.641 86	0.596 89	15
16	0.923 30	0.852 82	0.788 03	0.728 45	0.673 62	0.623 17	0.576 71	16
17	0.918 71	0.844 38	0.776 39	0.714 16	0.657 20	0.605 02	0.557 20	17
18	0.914 14	0.836 02	0.764 91	0.700 16	0.641 17	0.587 39	0.538 36	18
19	0.909 59	0.827 74	0.753 61	0.686 43	0.625 53	0.570 29	0.520 16	19
20	0.905 06	0.819 54	0.742 47	0.672 97	0.610 27	0.553 68	0.502 57	20
21	0.900 56	0.811 43	0.731 50	0.659 78	0.595 39	0.527 55	0.485 57	21
22	0.896 08	0.803 40	0.720 69	0.646 84	0.580 86	0.521 89	0.469 15	22
23	0.891 62	0.795 44	0.710 04	0.634 16	0.566 70	0.506 69	0.453 29	23
24	0.887 19	0.787 57	0.699 54	0.621 72	0.552 88	0.491 93	0.437 96	24
25	0.882 77	0.779 77	0.689 21	0.609 53	0.539 39	0.477 61	0.423 15	25
26	0.878 38	0.772 05	0.679 02	0.597 58	0.526 23	0.463 69	0.408 84	26
27	0.874 01	0.764 40	0.668 99	0.585 86	0.513 40	0.450 19	0.395 01	27
28	0.869 66	0.756 84	0.659 10	0.574 37	0.500 88	0.437 08	0.381 65	28
29	0.865 33	0.749 34	0.649 36	0.563 11	0.488 66	0.424 35	0.368 75	29
30	0.861 03	0.741 92	0.639 76	0.552 07	0.476 74	0.411 99	0.356 28	30
31	0.856 75	0.734 58	0.630 31	0.541 25	0.465 11	0.399 99	0.344 23	31
32	0.852 48	0.727 30	0.620 99	0.530 63	0.453 77	0.388 34	0.332 59	32
33	0.848 24	0.720 10	0.611 82	0.520 23	0.442 70	0.377 03	0.321 34	33
34	0.844 02	0.712 97	0.602 77	0.510 03	0.431 91	0.366 04	0.310 48	34
35	0.839 82	0.705 91	0.593 87	0.500 03	0.421 37	0.355 38	0.299 98	35
36	0.835 64	0.698 92	0.585 09	0.490 22	0.411 09	0.345 03	0.289 83	36
37	0.831 49	0.692 00	0.576 44	0.480 61	0.401 07	0.334 98	0.280 03	37
38	0.827 35	0.685 15	0.567 92	0.471 19	0.391 28	0.325 23	0.270 56	38
39	0.823 23	0.678 37	0.559 53	0.461 95	0.381 74	0.315 75	0.261 41	39
40	0.819 14	0.671 65	0.551 26	0.452 89	0.372 43	0.306 56	0.252 57	40

i n	4%	4½%	5%	5½%	6%	7%	8%	i n
1	0.961 54	0.956 94	0.952 38	0.947 87	0.943 40	0.934 58	0.925 93	1
2	0.924 56	0.915 73	0.907 03	0.898 45	0.890 00	0.873 44	0.857 34	2
3	0.889 00	0.876 30	0.863 84	0.851 61	0.839 62	0.816 30	0.793 83	3
4	0.854 80	0.838 56	0.822 70	0.807 22	0.792 09	0.762 90	0.735 03	4
5	0.821 93	0.802 45	0.783 53	0.765 13	0.747 26	0.712 99	0.680 58	5
6	0.790 31	0.767 90	0.746 22	0.725 25	0.704 96	0.666 34	0.630 17	6
7	0.759 92	0.734 83	0.710 68	0.687 44	0.665 06	0.622 75	0.583 49	7
8	0.730 69	0.703 19	0.676 84	0.651 60	0.627 41	0.582 01	0.540 27	8
9	0.702 59	0.672 90	0.644 61	0.617 63	0.591 90	0.543 93	0.500 25	9
10	0.675 56	0.643 93	0.613 91	0.585 43	0.558 39	0.508 35	0.463 19	10
11	0.649 58	0.616 20	0.584 68	0.554 91	0.526 79	0.475 09	0.428 88	11
12	0.624 60	0.589 66	0.556 84	0.525 98	0.496 97	0.444 01	0.397 11	12
13	0.600 57	0.564 27	0.530 32	0.498 56	0.468 84	0.414 96	0.367 70	13
14	0.577 48	0.539 97	0.505 07	0.472 57	0.442 30	0.387 82	0.340 46	14
15	0.555 26	0.516 72	0.481 02	0.447 93	0.417 27	0.362 45	0.315 24	15
16	0.533 91	0.494 47	0.458 11	0.424 58	0.393 65	0.338 73	0.291 89	16
17	0.513 37	0.473 18	0.436 30	0.402 45	0.371 36	0.316 57	0.270 27	17
18	0.493 63	0.452 80	0.415 52	0.381 47	0.350 34	0.295 86	0.250 25	18
19	0.474 64	0.433 30	0.395 73	0.361 58	0.330 51	0.276 51	0.231 71	19
20	0.456 39	0.414 64	0.376 89	0.342 73	0.311 80	0.258 42	0.214 55	20
21	0.438 83	0.396 79	0.358 94	0.324 86	0.294 16	0.241 51	0.198 66	21
22	0.421 96	0.379 70	0.341 85	0.307 93	0.277 51	0.225 71	0.183 94	22
23	0.405 73	0.363 35	0.325 57	0.291 87	0.261 80	0.210 95	0.170 32	23
24	0.390 12	0.347 70	0.310 07	0.276 66	0.246 98	0.197 15	0.157 70	24
25	0.375 12	0.332 73	0.295 30	0.262 23	0.233 00	0.184 25	0.146 02	25
26	0.360 69	0.318 40	0.281 24	0.248 56	0.219 81	0.172 20	0.135 20	26
27	0.346 82	0.304 69	0.267 85	0.235 60	0.207 37	0.160 93	0.125 19	27
28	0.333 48	0.291 57	0.255 09	0.223 32	0.195 63	0.150 40	0.115 91	28
29	0.320 65	0.279 02	0.242 95	0.211 68	0.184 56	0.140 56	0.107 33	29
30	0.308 32	0.267 00	0.231 38	0.200 64	0.174 11	0.131 37	0.099 38	30
31	0.296 46	0.255 50	0.220 36	0.190 18	0.164 25	0.122 77	0.092 02	31
32	0.285 06	0.244 50	0.209 87	0.180 27	0.154 96	0.114 74	0.085 20	32
33	0.274 09	0.233 97	0.199 87	0.170 87	0.146 19	0.107 23	0.078 89	33
34	0.263 55	0.223 90	0.190 35	0.161 96	0.137 91	0.100 22	0.073 05	34
35	0.253 42	0.214 25	0.181 29	0.153 52	0.130 11	0.093 66	0.067 63	35
36	0.243 67	0.205 03	0.172 66	0.145 52	0.122 74	0.087 54	0.062 62	36
37	0.234 30	0.196 20	0.164 44	0.137 93	0.115 79	0.081 81	0.057 99	37
38	0.225 29	0.187 75	0.156 61	0.130 74	0.109 24	0.076 46	0.053 69	38
39	0.216 62	0.179 67	0.149 15	0.123 92	0.103 06	0.071 46	0.049 71	39
40	0.208 29	0.171 93	0.142 05	0.117 46	0.097 22	0.066 78	0.046 03	40

Logarithms

	0	1	2	3	4	5	6	7	8	9	Differences								
											1	2	3	4	5	6	7	8	9
1.0	.0000	.0043	.0086	.0128	.0170	.0212	.0253	.0294	.0334	.0374	4	8	12	17	21	25	29	33	37
1.1	.0414	.0453	.0492	.0531	.0569	.0607	.0645	.0682	.0719	.0755	4	8	11	15	19	23	26	30	34
1.2	.0792	.0828	.0864	.0899	.0934	.0969	.1004	.1038	.1072	.1106	3	7	10	14	17	21	24	28	31
1.3	.1139	.1173	.1206	.1239	.1271	.1303	.1335	.1367	.1399	.1430	3	6	10	13	16	19	23	26	29
1.4	.1461	.1492	.1523	.1553	.1584	.1614	.1644	.1673	.1703	.1732	3	6	9	12	15	18	21	24	27
1.5	.1761	.1790	.1818	.1847	.1875	.1903	.1931	.1959	.1987	.2014	3	6	8	11	14	17	20	22	25
1.6	.2041	.2068	.2095	.2122	.2148	.2175	.2201	.2227	.2253	.2279	3	5	8	11	13	16	18	21	24
1.7	.2304	.2330	.2355	.2380	.2405	.2430	.2455	.2480	.2504	.2529	2	5	7	10	12	15	17	20	22
1.8	.2553	.2577	.2601	.2625	.2648	.2672	.2695	.2718	.2742	.2765	2	5	7	9	12	14	16	19	21
1.9	.2788	.2810	.2833	.2856	.2878	.2900	.2923	.2945	.2967	.2989	2	4	7	9	11	13	16	18	20
2.0	.3010	.3032	.3054	.3075	.3096	.3118	.3139	.3160	.3181	.3201	2	4	6	8	11	13	15	17	19
2.1	.3222	.3243	.3263	.3284	.3304	.3324	.3345	.3365	.3385	.3404	2	4	6	8	10	12	14	16	18
2.2	.3424	.3444	.3464	.3483	.3502	.3522	.3541	.3560	.3579	.3598	2	4	6	8	10	12	14	15	17
2.3	.3617	.3636	.3655	.3674	.3692	.3711	.3729	.3747	.3766	.3784	2	4	6	7	9	11	13	15	17
2.4	.3802	.3820	.3838	.3856	.3874	.3892	.3909	.3927	.3945	.3962	2	4	5	7	9	11	12	14	16
2.5	.3979	.3997	.4014	.4031	.4048	.4065	.4082	.4099	.4116	.4133	2	3	5	7	9	10	12	14	15
2.6	.4150	.4166	.4183	.4200	.4216	.4232	.4249	.4265	.4281	.4298	2	3	5	7	8	10	11	13	15
2.7	.4314	.4330	.4346	.4362	.4378	.4393	.4409	.4425	.4440	.4456	2	3	5	6	8	9	11	13	14
2.8	.4472	.4487	.4502	.4518	.4533	.4548	.4564	.4579	.4594	.4609	2	3	5	6	8	9	11	12	14
2.9	.4624	.4639	.4654	.4669	.4683	.4698	.4713	.4728	.4742	.4757	1	3	4	6	7	9	10	12	13
3.0	.4771	.4786	.4800	.4814	.4829	.4843	.4857	.4871	.4886	.4900	1	3	4	6	7	9	10	11	13
3.1	.4914	.4928	.4942	.4955	.4969	.4983	.4997	.5011	.5024	.5038	1	3	4	6	7	8	10	11	12
3.2	.5051	.5065	.5079	.5092	.5105	.5119	.5132	.5145	.5159	.5172	1	3	4	5	7	8	9	11	12
3.3	.5185	.5198	.5211	.5224	.5237	.5250	.5263	.5276	.5289	.5302	1	3	4	5	6	8	9	10	12
3.4	.5315	.5328	.5340	.5353	.5366	.5378	.5391	.5403	.5416	.5428	1	3	4	5	6	8	9	10	11
3.5	.5441	.5453	.5465	.5478	.5490	.5502	.5514	.5527	.5539	.5551	1	2	4	5	6	7	9	10	11
3.6	.5563	.5575	.5587	.5599	.5611	.5623	.5635	.5647	.5658	.5670	1	2	4	5	6	7	8	10	11
3.7	.5682	.5694	.5705	.5717	.5729	.5740	.5752	.5763	.5775	.5786	1	2	3	5	6	7	8	9	10
3.8	.5798	.5809	.5821	.5832	.5843	.5855	.5866	.5877	.5888	.5899	1	2	3	5	6	7	8	9	10
3.9	.5911	.5922	.5933	.5944	.5955	.5966	.5977	.5988	.5999	.6010	1	2	3	4	5	7	8	9	10
4.0	.6021	.6031	.6042	.6053	.6064	.6075	.6085	.6096	.6107	.6117	1	2	3	4	5	6	8	9	10
4.1	.6128	.6138	.6149	.6160	.6170	.6180	.6191	.6201	.6212	.6222	1	2	3	4	5	6	7	8	9
4.2	.6232	.6243	.6253	.6263	.6274	.6284	.6294	.6304	.6314	.6325	1	2	3	4	5	6	7	8	9
4.3	.6335	.6345	.6355	.6365	.6375	.6385	.6395	.6405	.6415	.6425	1	2	3	4	5	6	7	8	9
4.4	.6435	.6444	.6454	.6464	.6474	.6484	.6493	.6503	.6513	.6522	1	2	3	4	5	6	7	8	9
4.5	.6532	.6542	.6551	.6561	.6571	.6580	.6590	.6599	.6609	.6618	1	2	3	4	5	6	7	8	9
4.6	.6628	.6637	.6646	.6656	.6665	.6675	.6684	.6693	.6702	.6712	1	2	3	4	5	6	7	7	8
4.7	.6721	.6730	.6739	.6749	.6758	.6767	.6776	.6785	.6794	.6803	1	2	3	4	5	5	6	7	8
4.8	.6812	.6821	.6830	.6839	.6848	.6457	.6866	.6875	.6884	.6893	1	2	3	4	4	5	6	7	8
4.9	.6902	.6911	.6920	.6928	.6937	.6946	.6955	.6964	.6972	.6981	1	2	3	4	4	5	6	7	8
5.0	.6990	.6998	.7007	.7016	.7024	.7033	.7042	.7050	.7059	.7067	1	2	3	3	4	5	6	7	8
5.1	.7076	.7084	.7093	.7101	.7110	.7118	.7126	.7135	.7143	.7152	1	2	3	3	4	5	6	7	8
5.2	.7160	.7168	.7177	.7185	.7193	.7202	.7210	.7218	.7226	.7235	1	2	2	3	4	5	6	7	7
5.3	.7243	.7251	.7259	.7267	.7275	.7284	.7292	.7300	.7308	.7316	1	2	2	3	4	5	6	6	7
5.4	.7324	.7332	.7340	.7348	.7356	.7364	.7372	.7380	.7388	.7396	1	2	2	3	4	5	6	6	7
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

Logarithms

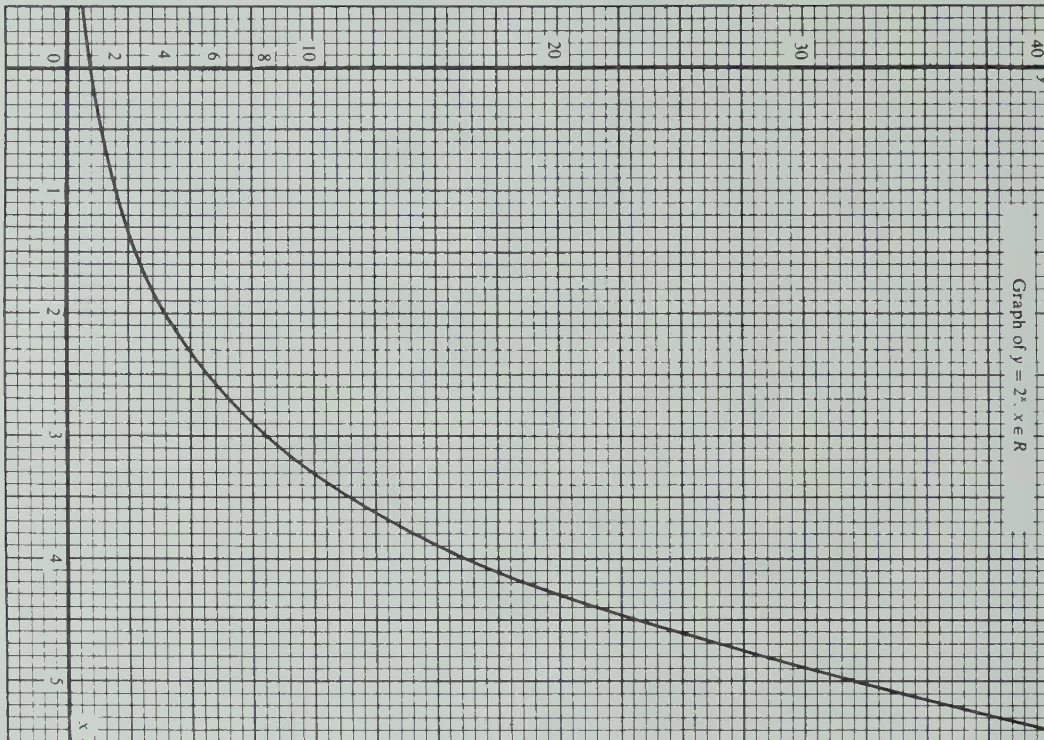
	0	1	2	3	4	5	6	7	8	9	Differences								
											1	2	3	4	5	6	7	8	9
5.5	.7404	.7412	.7419	.7427	.7435	.7443	.7451	.7459	.7466	.7474	1	2	2	3	4	5	5	6	7
5.6	.7482	.7490	.7497	.7505	.7513	.7520	.7528	.7536	.7543	.7551	1	2	2	3	4	5	5	6	7
5.7	.7559	.7566	.7574	.7582	.7589	.7597	.7604	.7612	.7619	.7627	1	2	2	3	4	5	5	6	7
5.8	.7634	.7642	.7649	.7657	.7664	.7672	.7679	.7686	.7694	.7701	1	1	2	3	4	4	5	6	7
5.9	.7709	.7716	.7723	.7731	.7738	.7745	.7752	.7760	.7767	.7774	1	1	2	3	4	4	5	6	7
6.0	.7782	.7789	.7796	.7803	.7810	.7818	.7825	.7832	.7839	.7846	1	1	2	3	4	4	5	6	6
6.1	.7853	.7860	.7868	.7875	.7882	.7889	.7896	.7903	.7910	.7917	1	1	2	3	4	4	5	6	6
6.2	.7924	.7931	.7938	.7945	.7952	.7959	.7966	.7973	.7980	.7987	1	1	2	3	3	4	5	6	6
6.3	.7993	.8000	.8007	.8014	.8021	.8028	.8035	.8041	.8048	.8055	1	1	2	3	3	4	5	5	6
6.4	.8062	.8069	.8075	.8082	.8089	.8096	.8102	.8109	.8116	.8122	1	1	2	3	3	4	5	5	6
6.5	.8129	.8135	.8142	.8149	.8156	.8162	.8169	.8176	.8182	.8189	1	1	2	3	3	4	5	5	6
6.6	.8195	.8202	.8209	.8215	.8222	.8228	.8235	.8241	.8248	.8254	1	1	2	3	3	4	5	5	6
6.7	.8261	.8267	.8274	.8280	.8287	.8293	.8299	.8306	.8312	.8319	1	1	2	3	3	4	5	5	6
6.8	.8325	.8331	.8338	.8344	.8351	.8357	.8363	.8370	.8376	.8382	1	1	2	3	3	4	4	5	6
6.9	.8388	.8395	.8401	.8407	.8414	.8420	.8426	.8432	.8439	.8445	1	1	2	2	3	4	4	5	6
7.0	.8451	.8457	.8463	.8470	.8476	.8482	.8488	.8494	.8500	.8506	1	1	2	2	3	4	4	5	6
7.1	.8513	.8519	.8525	.8531	.8537	.8543	.8549	.8555	.8561	.8567	1	1	2	2	3	4	4	5	5
7.2	.8573	.8579	.8585	.8591	.8597	.8603	.8609	.8615	.8621	.8627	1	1	2	2	3	4	4	5	5
7.3	.8633	.8639	.8645	.8651	.8657	.8663	.8669	.8675	.8681	.8686	1	1	2	2	3	4	4	5	5
7.4	.8692	.8698	.8704	.8710	.8716	.8722	.8727	.8733	.8739	.8745	1	1	2	2	3	4	4	5	5
7.5	.8751	.8756	.8762	.8768	.8774	.8779	.8785	.8791	.8797	.8802	1	1	2	2	3	3	4	5	5
7.6	.8808	.8814	.8820	.8825	.8831	.8837	.8842	.8848	.8854	.8859	1	1	2	2	3	3	4	5	5
7.7	.8865	.8871	.8876	.8882	.8887	.8893	.8899	.8904	.8910	.8915	1	1	2	2	3	3	4	4	5
7.8	.8921	.8927	.8932	.8938	.8943	.8949	.8954	.8960	.8965	.8971	1	1	2	2	3	3	4	4	5
7.9	.8976	.8982	.8987	.8993	.8998	.9004	.9009	.9015	.9020	.9025	1	1	2	2	3	3	4	4	5
8.0	.9031	.9036	.9042	.9047	.9053	.9058	.9063	.9069	.9074	.9079	1	1	2	2	3	3	4	4	5
8.1	.9085	.9090	.9096	.9101	.9106	.9112	.9117	.9122	.9128	.9133	1	1	2	2	3	3	4	4	5
8.2	.9138	.9143	.9149	.9154	.9159	.9165	.9170	.9175	.9180	.9186	1	1	2	2	3	3	4	4	5
8.3	.9191	.9196	.9201	.9206	.9212	.9217	.9222	.9227	.9232	.9238	1	1	2	2	3	3	4	4	5
8.4	.9243	.9248	.9253	.9258	.9263	.9269	.9274	.9279	.9284	.9289	1	1	2	2	3	3	4	4	5
8.5	.9294	.9299	.9304	.9309	.9315	.9320	.9325	.9330	.9335	.9340	1	1	2	2	3	3	4	4	5
8.6	.9345	.9350	.9355	.9360	.9365	.9370	.9375	.9380	.9385	.9390	1	1	2	2	3	3	4	4	5
8.7	.9395	.9400	.9405	.9410	.9415	.9420	.9425	.9430	.9435	.9440	0	1	1	2	2	3	3	4	4
8.8	.9445	.9450	.9455	.9460	.9465	.9469	.9474	.9479	.9484	.9489	0	1	1	2	2	3	3	4	4
8.9	.9494	.9499	.9504	.9509	.9513	.9518	.9523	.9528	.9533	.9538	0	1	1	2	2	3	3	4	4
9.0	.9542	.9547	.9552	.9557	.9562	.9566	.9571	.9576	.9581	.9586	0	1	1	2	2	3	3	4	4
9.1	.9590	.9595	.9600	.9605	.9609	.9614	.9619	.9624	.9628	.9633	0	1	1	2	2	3	3	4	4
9.2	.9638	.9643	.9647	.9652	.9657	.9661	.9666	.9671	.9675	.9680	0	1	1	2	2	3	3	4	4
9.3	.9685	.9689	.9694	.9699	.9703	.9708	.9713	.9717	.9722	.9727	0	1	1	2	2	3	3	3	4
9.4	.9631	.9736	.9741	.9745	.9750	.9754	.9759	.9763	.9768	.9773	0	1	1	2	2	3	3	4	4
9.5	.9777	.9782	.9786	.9791	.9795	.9800	.9805	.9809	.9814	.9818	0	1	1	2	2	3	3	4	4
9.6	.9823	.9827	.9832	.9836	.9841	.9845	.9850	.9854	.9859	.9863	0	1	1	2	2	3	3	4	4
9.7	.9868	.9872	.9877	.9881	.9886	.9890	.9894	.9899	.9903	.9908	0	1	1	2	2	3	3	4	4
9.8	.9912	.9917	.9921	.9926	.9930	.9934	.9939	.9943	.9948	.9952	0	1	1	2	2	3	3	4	4
9.9	.9956	.9961	.9965	.9969	.9974	.9978	.9983	.9987	.9991	.9996	0	1	1	2	2	3	3	3	4
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

VALUES OF THE EXPONENTIAL FUNCTION $y = 10^x$

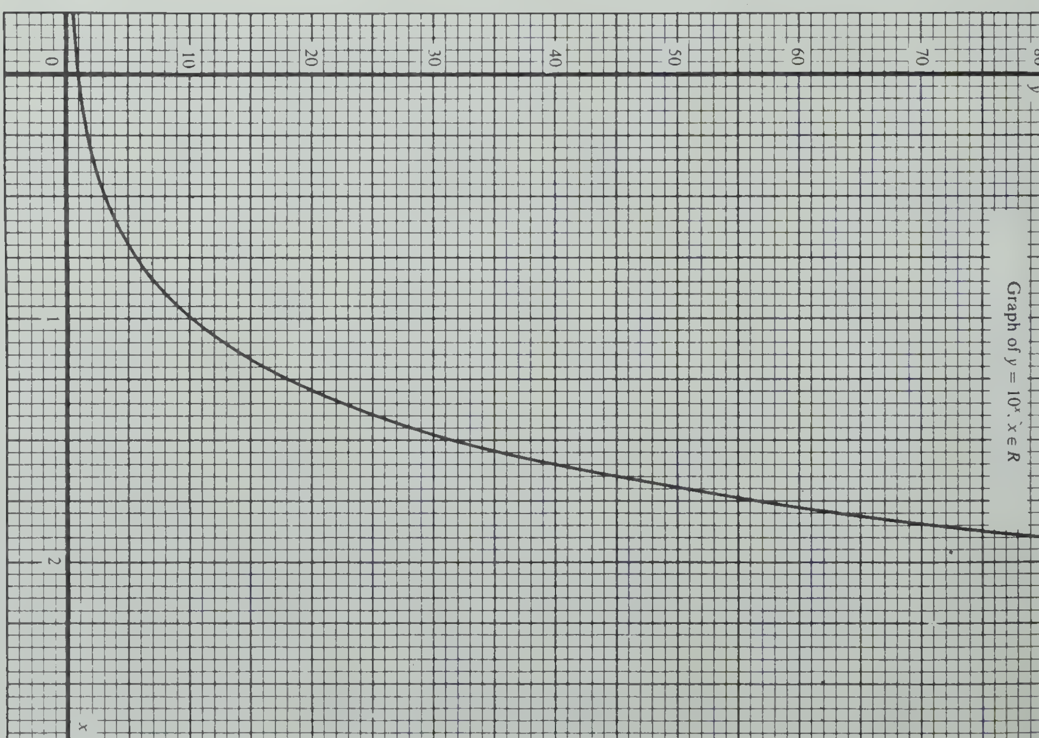
														Differences				
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0.00	1.000	1.002	1.005	1.007	1.009	1.012	1.014	1.016	1.019	1.021	0	0	1	1	1	1	2	2	2
0.01	1.023	1.026	1.028	1.030	1.033	1.035	1.038	1.040	1.042	1.045	0	0	1	1	1	1	2	2	2
0.02	1.047	1.050	1.052	1.054	1.057	1.059	1.062	1.064	1.067	1.069	0	0	1	1	1	1	2	2	2
0.03	1.072	1.074	1.074	1.079	1.081	1.084	1.086	1.089	1.091	1.094	0	0	1	1	1	1	2	2	2
0.04	1.096	1.099	1.102	1.104	1.107	1.109	1.112	1.114	1.117	1.119	0	1	1	1	1	2	2	2	2
0.05	1.122	1.125	1.127	1.130	1.132	1.135	1.138	1.140	1.143	1.146	0	1	1	1	1	2	2	2	2
0.06	1.148	1.151	1.153	1.156	1.159	1.161	1.164	1.167	1.169	1.172	0	1	1	1	1	2	2	2	2
0.07	1.175	1.178	1.180	1.183	1.186	1.189	1.191	1.194	1.197	1.199	0	1	1	1	1	2	2	2	2
0.08	1.202	1.205	1.208	1.211	1.213	1.216	1.219	1.222	1.225	1.227	0	1	1	1	1	2	2	2	3
0.09	1.230	1.233	1.236	1.239	1.242	1.245	1.247	1.250	1.253	1.256	0	1	1	1	1	2	2	2	3
0.10	1.259	1.262	1.265	1.268	1.271	1.274	1.276	1.279	1.282	1.285	0	1	1	1	1	2	2	2	3
0.11	1.288	1.291	1.294	1.297	1.300	1.303	1.306	1.309	1.312	1.315	0	1	1	1	2	2	2	2	3
0.12	1.318	1.321	1.324	1.327	1.330	1.334	1.337	1.340	1.343	1.346	0	1	1	1	2	2	2	2	3
0.13	1.349	1.352	1.355	1.358	1.361	1.365	1.368	1.371	1.374	1.377	0	1	1	1	2	2	2	3	3
0.14	1.380	1.384	1.387	1.390	1.393	1.396	1.400	1.403	1.406	1.409	0	1	1	1	2	2	2	3	3
0.15	1.413	1.416	1.419	1.422	1.426	1.429	1.432	1.435	1.439	1.442	0	1	1	1	2	2	2	3	3
0.16	1.445	1.449	1.452	1.455	1.459	1.462	1.466	1.469	1.472	1.476	0	1	1	1	2	2	2	3	3
0.17	1.479	1.483	1.486	1.489	1.493	1.496	1.500	1.503	1.507	1.510	0	1	1	1	2	2	2	3	3
0.18	1.514	1.517	1.521	1.524	1.528	1.531	1.535	1.538	1.542	1.545	0	1	1	1	2	2	2	3	3
0.19	1.549	1.552	1.556	1.560	1.563	1.567	1.570	1.574	1.578	1.581	0	1	1	1	2	2	3	3	3
0.20	1.585	1.589	1.592	1.596	1.600	1.603	1.607	1.611	1.614	1.618	0	1	1	1	2	2	3	3	3
0.21	1.622	1.626	1.629	1.633	1.637	1.641	1.644	1.648	1.652	1.656	0	1	1	2	2	2	3	3	3
0.22	1.660	1.663	1.667	1.671	1.675	1.679	1.683	1.687	1.690	1.694	0	1	1	2	2	2	3	3	3
0.23	1.698	1.702	1.706	1.710	1.714	1.718	1.722	1.726	1.730	1.734	0	1	1	2	2	2	3	3	4
0.24	1.738	1.742	1.746	1.750	1.754	1.758	1.762	1.766	1.770	1.774	0	1	1	2	2	2	3	3	4
0.25	1.778	1.782	1.786	1.791	1.795	1.799	1.803	1.807	1.811	1.816	0	1	1	2	2	2	3	3	4
0.26	1.820	1.824	1.828	1.832	1.837	1.841	1.845	1.849	1.854	1.858	0	1	1	2	2	3	3	3	4
0.27	1.862	1.866	1.871	1.875	1.879	1.884	1.888	1.892	1.897	1.901	0	1	1	2	2	3	3	3	4
0.28	1.905	1.910	1.914	1.919	1.923	1.928	1.932	1.936	1.941	1.945	0	1	1	2	2	3	3	4	4
0.29	1.950	1.954	1.959	1.963	1.968	1.972	1.977	1.982	1.986	1.991	0	1	1	2	2	3	3	4	4
0.30	1.995	2.000	2.004	2.009	2.014	2.018	2.023	2.028	2.032	2.037	0	1	1	2	2	3	3	4	4
0.31	2.042	2.046	2.051	2.056	2.061	2.065	2.070	2.075	2.080	2.084	0	1	1	2	2	3	3	4	4
0.32	2.089	2.094	2.099	2.104	2.109	2.113	2.118	2.123	2.128	2.133	0	1	1	2	2	3	3	4	4
0.33	2.138	2.143	2.148	2.153	2.158	2.163	2.168	2.173	2.178	2.183	0	1	1	2	2	3	3	4	4
0.34	2.188	2.193	2.198	2.203	2.208	2.213	2.218	2.223	2.228	2.234	1	1	2	2	3	3	4	4	5
0.35	2.239	2.244	2.249	2.254	2.259	2.265	2.270	2.275	2.280	2.286	1	1	2	2	3	3	4	4	5
0.36	2.291	2.296	2.301	2.307	2.312	2.317	2.323	2.328	2.333	2.339	1	1	2	2	3	3	4	4	5
0.37	2.344	2.350	2.355	2.360	2.366	2.371	2.377	2.382	2.388	2.393	1	1	2	2	3	3	4	4	5
0.38	2.399	2.404	2.410	2.415	2.421	2.427	2.432	2.438	2.443	2.449	1	1	2	2	3	3	4	4	5
0.39	2.455	2.460	2.466	2.472	2.477	2.483	2.489	2.495	2.500	2.506	1	1	2	2	3	3	4	5	5
0.40	2.512	2.518	2.524	2.529	2.535	2.541	2.547	2.553	2.559	2.564	1	1	2	2	3	4	4	5	5
0.41	2.570	2.576	2.582	2.588	2.594	2.600	2.606	2.612	2.618	2.624	1	1	2	2	3	4	4	5	5
0.42	2.630	2.636	2.642	2.649	2.655	2.661	2.667	2.673	2.679	2.685	1	1	2	2	3	4	4	5	6
0.43	2.692	2.698	2.704	2.710	2.716	2.723	2.729	2.735	2.742	2.748	1	1	2	3	3	4	4	5	6
0.44	2.754	2.761	2.767	2.773	2.780	2.786	2.793	2.799	2.805	2.812	1	1	2	3	3	4	4	5	6
0.45	2.818	2.825	2.831	2.838	2.844	2.851	2.858	2.864	2.871	2.877	1	1	2	3	3	4	5	5	6
0.46	2.884	2.891	2.897	2.904	2.911	2.917	2.924	2.931	2.938	2.944	1	1	2	3	3	4	5	5	6
0.47	2.951	2.958	2.965	2.972	2.979	2.985	2.992	2.999	3.006	3.013	1	1	2	3	3	4	5	5	6
0.48	3.020	3.027	3.034	3.041	3.048	3.055	3.062	3.069	3.076	3.083	1	1	2	3	4	4	5	6	6
0.49	3.090	3.097	3.105	3.112	3.119	3.126	3.133	3.141	3.148	3.155	1	1	2	3	4	4	5	6	6

VALUES OF THE EXPONENTIAL FUNCTION $y = 10^x$

											Differences								
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0.50	3.162	3.170	3.177	3.184	3.192	3.199	3.206	3.214	3.221	3.228	1	2	3	3	4	4	5	6	7
0.51	3.236	3.243	3.251	3.258	3.266	3.273	3.281	3.289	3.296	3.304	1	2	2	3	4	5	5	6	7
0.52	3.311	3.319	3.327	3.334	3.342	3.350	3.357	3.365	3.373	3.381	1	2	2	3	4	5	5	6	7
0.53	3.388	3.396	3.404	3.412	3.420	3.428	3.436	3.443	3.451	3.459	1	2	2	3	4	5	6	6	7
0.54	3.467	3.475	3.483	3.491	3.499	3.508	3.516	3.524	3.532	3.540	1	2	2	3	4	5	6	6	7
0.55	3.548	3.556	3.565	3.573	3.581	3.589	3.597	3.606	3.614	3.622	1	2	2	3	4	5	6	7	7
0.56	3.631	3.639	3.648	3.656	3.664	3.673	3.681	3.690	3.698	3.707	1	2	3	3	4	5	6	7	8
0.57	3.715	3.724	3.733	3.741	3.750	3.758	3.767	3.776	3.784	3.793	1	2	3	3	4	5	6	7	8
0.58	3.802	3.811	3.819	3.828	3.837	3.846	3.855	3.864	3.873	3.882	1	2	3	4	4	5	6	7	8
0.59	3.890	3.899	3.908	3.917	3.926	3.936	3.945	3.954	3.963	3.972	1	2	3	4	5	5	6	7	8
0.60	3.981	3.990	3.999	4.009	4.018	4.027	4.036	4.046	4.055	4.064	1	2	3	4	5	6	6	7	8
0.61	4.074	4.083	4.093	4.102	4.111	4.121	4.130	4.140	4.150	4.159	1	2	3	4	5	6	7	8	9
0.62	4.169	4.178	4.188	4.198	4.207	4.217	4.227	4.236	4.246	4.256	1	2	3	4	5	6	7	8	9
0.63	4.266	4.276	4.285	4.295	4.305	4.315	4.325	4.335	4.345	4.355	1	2	3	4	5	6	7	8	9
0.64	4.365	4.375	4.385	4.395	4.406	4.416	4.426	4.436	4.446	4.457	1	2	3	4	5	6	7	8	9
0.65	4.467	4.477	4.487	4.498	4.508	4.519	5.529	4.539	4.550	4.560	1	2	3	4	5	6	7	8	9
0.66	4.571	4.581	4.592	4.603	4.613	4.624	4.634	4.645	4.656	4.667	1	2	3	4	5	6	7	9	10
0.67	4.677	4.688	4.699	4.710	4.721	4.732	4.742	4.753	4.764	4.775	1	2	3	4	5	7	8	9	10
0.68	4.786	4.797	4.808	4.819	4.831	4.842	4.853	4.864	4.875	4.887	1	2	3	4	6	7	8	9	10
0.69	4.898	4.909	4.920	4.932	4.943	4.955	4.966	4.977	4.989	5.000	1	2	3	5	6	7	8	9	10
0.70	5.012	5.023	5.035	5.047	5.058	5.070	5.082	5.093	5.105	5.117	1	2	4	5	6	7	8	9	11
0.71	5.129	5.140	5.152	5.164	5.176	5.188	5.200	5.212	5.224	5.236	1	2	4	5	6	7	8	10	11
0.72	5.248	5.260	5.272	5.284	5.297	5.309	5.321	5.333	5.346	5.358	1	2	4	5	6	7	9	10	11
0.73	5.370	5.383	5.395	5.408	5.420	5.433	5.445	5.458	5.470	5.483	1	3	4	5	6	8	9	10	11
0.74	5.495	5.508	5.521	5.534	5.546	5.559	5.572	5.585	5.598	5.610	1	3	4	5	6	8	9	10	12
0.75	5.623	5.636	5.649	5.662	5.675	5.689	5.702	5.715	5.728	5.741	1	3	4	5	7	8	9	10	12
0.76	5.754	5.768	5.781	5.794	5.808	5.821	5.834	5.848	5.861	5.875	1	3	4	5	7	8	9	11	12
0.77	5.888	5.902	5.916	5.929	5.943	5.957	5.970	5.984	5.998	6.012	1	3	4	5	7	8	10	11	12
0.78	6.026	6.039	6.053	6.067	6.081	6.095	6.109	6.124	6.138	6.152	1	3	4	6	7	8	10	11	13
0.79	6.166	6.180	6.194	6.209	6.223	6.236	6.252	6.266	6.281	6.295	1	3	4	6	7	9	10	11	13
0.80	6.310	6.324	6.339	6.353	6.368	6.383	6.397	6.412	6.427	6.442	1	3	4	6	7	9	10	12	13
0.81	6.457	6.471	6.486	6.501	6.516	6.531	6.546	6.561	6.577	6.592	2	3	5	6	8	9	11	12	14
0.82	6.607	6.622	6.637	6.653	6.668	6.683	6.699	6.714	6.730	6.745	2	3	5	6	8	9	11	12	14
0.83	6.761	6.776	6.792	6.808	6.823	6.839	6.855	6.871	6.887	6.902	2	3	5	6	8	9	11	13	14
0.84	6.918	6.934	6.950	6.966	6.982	6.998	7.015	7.031	7.047	7.063	2	3	5	6	8	10	11	13	15
0.85	7.079	7.096	7.112	7.129	7.145	7.161	7.178	7.194	7.211	7.228	2	3	5	7	8	10	12	13	15
0.86	7.244	7.261	7.278	7.295	7.311	7.328	7.345	7.362	7.379	7.396	2	3	5	7	8	10	12	13	15
0.87	7.413	7.430	7.447	7.464	7.482	7.499	7.516	7.534	7.551	7.568	2	3	5	7	9	10	12	14	16
0.88	7.586	7.603	7.621	7.638	7.656	7.674	7.691	7.709	7.727	7.745	2	4	5	7	9	11	12	14	16
0.89	7.762	7.780	7.798	7.816	7.834	7.852	7.870	7.889	7.907	7.925	2	4	5	7	9	11	13	14	16
0.90	7.943	7.962	7.980	7.998	8.017	8.035	8.054	8.072	8.091	8.110	2	4	6	7	9	11	13	15	17
0.91	8.128	8.147	8.166	8.185	8.204	8.222	8.241	8.260	8.279	8.299	2	4	6	8	9	11	13	15	17
0.92	8.318	8.337	8.356	8.375	8.395	8.414	8.433	8.453	8.472	8.492	2	4	6	8	10	12	14	15	17
0.93	8.511	8.531	8.551	8.570	8.590	8.610	8.630	8.650	8.670	8.690	2	4	6	8	10	12	14	16	18
0.94	8.710	8.730	8.750	8.770	8.790	8.810	8.831	8.851	8.872	8.892	2	4	6	8	10	12	14	16	18
0.95	8.913	8.933	8.954	8.974	8.995	9.016	9.036	9.057	9.078	9.099	2	4	6	8	10	12	15	17	19
0.96	9.120	9.141	9.162	9.183	9.204	9.226	9.247	9.268	9.290	9.311	2	4	6	8	11	13	15	17	19
0.97	9.333	9.354	9.376	9.397	9.419	9.441	9.462	9.484	9.506	9.528	2	4	7	9	11	13	15	17	20
0.98	9.550	9.572	9.594	9.616	9.638	9.661	9.683	9.705	9.727	9.750	2	4	7	9	11	13	16	18	20
0.99	9.772	9.795	9.817	9.840	9.863	9.886	9.908	9.931	9.954	9.977	2	5	7	9	11	14	16	18	20



Graph of $y = 2^x$, $x \in R$



Graph of $y = 10^x$, $x \in R$

INDEX

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